

**1. Gaussian Elimination**

**Learning Goal:** The goal of this problem is to use Gaussian Elimination to describe solutions to systems, both qualitatively and quantitatively. Please review **Note 1B** to understand this problem better.

Write each system as an augmented matrix, and then solve using Gaussian Elimination. Also determine whether each system has no solution, a unique solution, or a set of infinitely many solutions.

(a) Solve the following system of equations:

$$\begin{aligned}x_1 - x_2 + 2x_3 &= 15 \\3x_2 - x_3 &= 8 \\x_1 + 2x_3 &= 21\end{aligned}$$

**Answer:** The system can be written in matrix-vector form:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 21 \end{bmatrix}$$

Then, we write the system as an augmented matrix:

$$\left[ \begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 3 & -1 & 8 \\ 1 & 0 & 2 & 21 \end{array} \right]$$

Here is an example of one possible route taken by Gaussian Elimination (there are many different ways to perform the algorithm):

$$\begin{aligned} \left[ \begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 3 & -1 & 8 \\ 1 & 0 & 2 & 21 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 3 & -1 & 8 \\ 0 & 1 & 0 & 6 \end{array} \right] \text{ using } R_3 \leftarrow R_3 - R_1 \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 1 & 0 & 6 \\ 0 & 3 & -1 & 8 \end{array} \right] \text{ swapping } R_2 \text{ and } R_3 \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -1 & -10 \end{array} \right] \text{ using } R_3 \leftarrow R_3 - 3R_2 \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 10 \end{array} \right] \text{ using } R_3 \leftarrow -R_3 \end{aligned}$$

The system is now in **upper triangular matrix** form (row echelon form), and we can see that all three columns have pivot elements of "1" (leading entries 1's). So we can determine that  $x_1$ ,  $x_2$ , and  $x_3$  are all **basic variables**. With three nonzero, consistent equations in this form, the system must have a **unique solution**.

We can either **back-substitute** or continue applying row reductions to generate a **reduced row-echelon form**.

**Method I: Back-Substitution:** From our augmented matrix in upper triangular form, we have the equations

$$\begin{aligned}x_1 - x_2 + 2x_3 &= 15 \\x_2 &= 6 \\x_3 &= 10\end{aligned}$$

The latter two already tell us what  $x_2$  and  $x_3$  will be, so we can plug in these values into the first equation to find  $x_1 = 15 + 6 - 2(10) = 1$ . So the unique solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 10 \end{bmatrix}$$

**Method II: Reduced Row-Echelon Form:** We continue row reduction to eliminate the non-zero elements *above* the pivots:

$$\begin{aligned}\left[ \begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 10 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 21 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 10 \end{array} \right] \text{ using } R_1 \leftarrow R_1 + R_2 \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 10 \end{array} \right] \text{ using } R_1 \leftarrow R_1 - 2R_3\end{aligned}$$

From our augmented matrix in reduced row echelon form, we have the equations:

$$\begin{aligned}x_1 &= 1 \\x_2 &= 6 \\x_3 &= 10\end{aligned}$$

So we get the same solution as Method I.

(b) [WALK-THROUGH] Solve the following system of equations:

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 4 \\x_1 + x_3 &= 0 \\-2x_1 + 2x_2 + x_3 &= 5 \\x_1 + x_2 + 2x_3 &= 2\end{aligned}$$

**Answer:** The matrix-vector form can be written as:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 2 \end{bmatrix}$$

We write the system as an augmented matrix:

$$\begin{aligned}
 \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ -2 & 2 & 1 & 5 \\ 1 & 1 & 2 & 2 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & 6 & 7 & 13 \\ 0 & -1 & -1 & -2 \end{array} \right] &\text{using } R_2 \leftarrow R_2 - R_1; R_3 \leftarrow R_3 + 2R_1; R_4 \leftarrow R_4 - R_1 \\
 &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 6 & 7 & 13 \\ 0 & -1 & -1 & -2 \end{array} \right] &\text{using } R_2 \leftarrow -R_2/2 \\
 &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] &\text{using } R_3 \leftarrow R_3 - 6R_2; R_4 \leftarrow R_4 + R_2 \\
 &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] &\text{using } R_1 \leftarrow R_1 - 2R_2 \\
 &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] &\text{using } R_1 \leftarrow R_1 - R_3; R_2 \leftarrow R_2 - R_3;
 \end{aligned}$$

The system is now in **upper triangular matrix** form, and we can see that all three columns have pivot elements of "1" (leading entries 1's). So we can determine that  $x_1$ ,  $x_2$ , and  $x_3$  are all **basic variables**, so there is a **unique solution**.

From the first three rows we get:

$$\begin{aligned}
 x_1 &= -1 \\
 x_2 &= 1 \\
 x_3 &= 1
 \end{aligned}$$

So we can write the solution set as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(c) [WALK-THROUGH]

(i) Now let us just change the third equation from the last problem to

$$-2x_1 + 2x_2 = 4.$$

The other three equations are unchanged. Do you still have a unique solution?

(ii) What if you change the third equation to

$$-2x_1 + 2x_2 = 5?$$

**Answer:** (i) We write the system as an augmented matrix:

$$\begin{aligned}
 \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ -2 & 2 & 0 & 4 \\ 1 & 1 & 2 & 2 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & 6 & 6 & 12 \\ 0 & -1 & -1 & -2 \end{array} \right] &\text{using } R_2 \leftarrow R_2 - R_1; R_3 \leftarrow R_3 + 2R_1; R_4 \leftarrow R_4 - R_1 \\
 &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 6 & 6 & 12 \\ 0 & -1 & -1 & -2 \end{array} \right] &\text{using } R_2 \leftarrow -R_2/2 \\
 &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] &\text{using } R_3 \leftarrow R_3 - 6R_2; R_4 \leftarrow R_4 + R_2 \\
 &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] &\text{using } R_1 \leftarrow R_1 - 2R_2
 \end{aligned}$$

After Gaussian Elimination, we get two rows that look like  $\left[ \begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array} \right]$ . The system is determined by the other two rows, which represent the equations  $x_1 + x_3 = 0$  and  $x_2 + x_3 = 2$ . Since we have two non-zero rows and three variables, there are **infinitely many solutions**.

The first two columns contain leading 1's or pivot elements. So  $x_1$  and  $x_2$  should be **basic variables**. That leaves  $x_3$  to be the **free variable**.

From the first two rows we get:

$$\begin{aligned}
 x_1 + x_3 = 0 &\implies x_1 = -x_3 \\
 x_2 + x_3 = 2 &\implies x_2 = 2 - x_3
 \end{aligned}$$

So we can write the solution set as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 2 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

**Alternative approach:** We can choose a **parametric representation** of solutions. We choose the free variable as:  $x_3 = a$ , where  $a$  is any scalar. Again the first two rows give us the following:

$$\begin{aligned}
 x_1 + a = 0 &\implies x_1 = -a \\
 x_2 + a = 2 &\implies x_2 = 2 - a
 \end{aligned}$$

So we can write the solution set as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a \\ 2 - a \\ a \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} a + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Either approach to representing the solution set is fine.

(ii) We write the system as an augmented matrix:

$$\begin{aligned} \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ -2 & 2 & 0 & 5 \\ 1 & 1 & 2 & 2 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & 6 & 6 & 13 \\ 0 & -1 & -1 & -2 \end{array} \right] \text{ using } R_2 \leftarrow R_2 - R_1; R_3 \leftarrow R_3 + 2R_1; R_4 \leftarrow R_4 - R_1 \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 6 & 6 & 13 \\ 0 & -1 & -1 & -2 \end{array} \right] \text{ using } R_2 \leftarrow -R_2/2 \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \text{ using } R_3 \leftarrow R_3 - 6R_2; R_4 \leftarrow R_4 + R_2 \end{aligned}$$

This system has **no solution**. The third row is  $[0 \ 0 \ 0 \ | \ a]$ , where  $a \neq 0$  in the augmented matrix. This row corresponds to the equation  $0x_1 + 0x_2 + 0x_3 = 1$ , which cannot be valid. This means the measurements are not correct, i.e. the system is inconsistent.

(d) (PRACTICE) Solve the following system of equations:

$$\begin{aligned} 2x_2 + 4x_3 &= -2 \\ -5x_3 &= 10 \\ x_1 + x_2 - 3x_3 &= 8 \end{aligned}$$

**Answer:** The matrix-vector form of the system is:

$$\begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & -5 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 8 \end{bmatrix}$$

Then we write the system as an augmented matrix:

$$\left[ \begin{array}{ccc|c} 0 & 2 & 4 & -2 \\ 0 & 0 & -5 & 10 \\ 1 & 1 & -3 & 8 \end{array} \right]$$

Notice that the first row already has one 0, the second row has two 0's, and the third row has a nonzero element in the first column. If we swap rows around, this should get us our upper triangular matrix:

$$\begin{aligned} \left[ \begin{array}{ccc|c} 0 & 2 & 4 & -2 \\ 0 & 0 & -5 & 10 \\ 1 & 1 & -3 & 8 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 0 & 2 & 4 & -2 \\ 1 & 1 & -3 & 8 \\ 0 & 0 & -5 & 10 \end{array} \right] \text{ swapping } R_2 \text{ and } R_3 \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & -3 & 8 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & -5 & 10 \end{array} \right] \text{ swapping } R_1 \text{ and } R_2 \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & -3 & 8 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array} \right] R_2 \leftarrow R_2/2 \text{ and } R_3 \leftarrow R_3/-5 \end{aligned}$$

We've reached upper triangular matrix form, and there are three equations with three basic variables, indicating the existence of a **unique solution**.

We can either use back-substitution or further row reduction to find the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

(e) (PRACTICE) Solve the following system of equations:

$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= -3 \\ 2x_1 + 6x_2 - 4x_3 &= -5 \end{aligned}$$

**Answer:** The equations can be written in matrix-vector form

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

Then we can write the augmented matrix form:

$$\left[ \begin{array}{ccc|c} 1 & 3 & -2 & -3 \\ 2 & 6 & -4 & -5 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 3 & -2 & -3 \\ 0 & 0 & 0 & 1 \end{array} \right] \text{ using } R_2 \leftarrow R_2 - 2R_1$$

This system has **no solution**. The last row is  $[0 \ 0 \ 0 \ | \ a]$ , where  $a \neq 0$  in the augmented matrix. This row corresponds to the equation  $0x_1 + 0x_2 + 0x_3 = 1$ , which cannot be valid. This means the measurements are not correct, i.e. the system is inconsistent.

## 2. Computations: Matrix-Vector Operations [WALKTHROUGH]

**Learning Goal:** The goal of this problem is to present various cases of matrix-vector operations such as addition, multiplication, and transpose. Please review [Note 2A: Section 2.3](#) and [Note 2B: Section 2.1](#) to understand this problem better.

Consider the following matrices and vectors. Complete the parts below.

$$A = \begin{bmatrix} 2 & 4 \\ 5 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

(a) What is the transpose of  $\vec{v}$ ?

**Answer:**

$$\vec{v}^T = [1 \ 2 \ 3]$$

(b) What is  $(\vec{v} + \vec{w})^T$ ? Find  $\vec{v}^T + \vec{w}^T$  too. Compare the results.

**Answer:**

$$\vec{v} + \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 2+(-1) \\ 3+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

So  $(\vec{v} + \vec{w})^T = [1 \ 1 \ 7]$ .

For the second part we have:

$$\begin{aligned}\vec{v}^T &= [1 \ 2 \ 3] \\ \vec{w}^T &= [0 \ -1 \ 4]\end{aligned}$$

Hence  $\vec{v}^T + \vec{w}^T = [1 \ 1 \ 7]$ . From these results we can see that:  $(\vec{v} + \vec{w})^T = \vec{v}^T + \vec{w}^T$

(c) What is  $2\vec{v} - 4\vec{w}$ ?

**Answer:**

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(1) \\ 2(2) \\ 2(3) \end{bmatrix} - \begin{bmatrix} 4(0) \\ 4(-1) \\ 4(4) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ -4 \\ 16 \end{bmatrix} = \begin{bmatrix} 2-0 \\ 4+4 \\ 6-16 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -10 \end{bmatrix}$$

(d) What is  $\vec{v}^T \vec{w}$ ?

**Answer:** The dimension of  $\vec{v}^T$  is  $1 \times 3$  and the dimension of  $\vec{w}$  is  $3 \times 1$ . Since the product of an  $m \times n$  and an  $n \times p$  matrix results in an  $m \times p$  matrix, the output in this part is going to be a scalar (i.e.  $1 \times 1$ ).

$$[1 \ 2 \ 3] \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = [(1)(0) + (2)(-1) + (3)(4)] = [0 - 2 + 12] = [10]$$

(e) What is  $A\vec{u}_1$ ? What is  $A\vec{u}_2$ ?

**Answer:** The dimension of  $\mathbf{A}$  is  $2 \times 2$  and the dimension of  $\vec{u}_1$  is  $2 \times 1$ . Therefore, the output in this part is going to have a dimension of  $2 \times 1$ .

$$\begin{aligned}\mathbf{A}\vec{u}_1 &= \begin{bmatrix} 2 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} (2)(1) + (4)(2) \\ (5)(1) + (-3)(2) \end{bmatrix} = \begin{bmatrix} 2+8 \\ 5-6 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix} \\ \mathbf{A}\vec{u}_2 &= \begin{bmatrix} 2 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} (2)(3) + (4)(-4) \\ (5)(3) + (-3)(-4) \end{bmatrix} = \begin{bmatrix} 6-16 \\ 15+12 \end{bmatrix} = \begin{bmatrix} -10 \\ 27 \end{bmatrix}\end{aligned}$$

(f) What is  $\mathbf{AB}$ ? (Do the columns of  $\mathbf{AB}$  look familiar?)

**Answer:** The dimension of both  $\mathbf{A}$  and  $\mathbf{B}$  is  $2 \times 2$ . Therefore, the output in this part is going to have a dimension of  $2 \times 2$ .

$$\begin{bmatrix} 2 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} (2)(1) + (4)(2) & (2)(3) + (4)(-4) \\ (5)(1) + (-3)(2) & (5)(3) + (-3)(-4) \end{bmatrix} = \begin{bmatrix} 2+8 & 6-16 \\ 5-6 & 15+12 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ -1 & 27 \end{bmatrix}$$

We can observe that the columns of  $\mathbf{AB}$  are the same as the results from the last part. This is because the columns of matrix  $\mathbf{B}$  are the same as vectors  $\vec{u}_1$  and  $\vec{u}_2$ , i.e.

$$\begin{aligned}\mathbf{B} &= \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \\ \mathbf{AB} &= \mathbf{A} [\vec{u}_1 \ \vec{u}_2] = [\mathbf{A}\vec{u}_1 \ \mathbf{A}\vec{u}_2]\end{aligned}$$

(g) Find  $\mathbf{B}^T$ . Then express  $\mathbf{B}^T$  in terms of  $\vec{u}_1$  and  $\vec{u}_2$ ?

**Answer:**

$$\mathbf{B}^T = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

$$\mathbf{B}^T = [\vec{u}_1 \quad \vec{u}_2]^T = \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \end{bmatrix}$$

### 3. Linear or Nonlinear

**Learning Goal:** The goal of this problem is to draw a distinction between linear and non-linear functions. Please review [Section 1.4 of Note 1](#) to understand this problem better.

Determine whether the following functions are linear or nonlinear.

(a) [WALK-THROUGH]

$$f(x_1, x_2) = 3x_1 + 4x_2$$

**Answer:** To check for linearity, check for **superposition** (additivity) and **homogeneity** (multiplicative scaling). In other words we must check that:  $f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2)$ , where,  $\alpha$  and  $\beta$  are scalars.

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= 3(\alpha x_1 + \beta y_1) + 4(\alpha x_2 + \beta y_2) \\ &= \alpha(3x_1 + 4x_2) + \beta(3y_1 + 4y_2) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

So this function is **linear**. Alternatively you can state that this function is linear because it is of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where  $a_1$  and  $a_2$  are constants.

(b)

$$f(x_1, x_2) = x_1^2 + e^{x_2}$$

**Answer:** We check again for superposition and homogeneity.

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= (\alpha x_1 + \beta y_1)^2 + e^{\alpha x_2 + \beta y_2} \\ &= (\alpha x_1)^2 + (\beta y_1)^2 + 2\alpha\beta x_1 y_1 + e^{\alpha x_2} e^{\beta y_2} \\ &\neq \alpha(x_1^2 + e^{x_2}) + \beta(y_1^2 + e^{y_2}) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Hence this function is **not linear**.

(c)

$$f(x_1, x_2) = \sin(a)x_1 + e^b x_2,$$

where  $a$  and  $b$  are constants.



**Answer:** Check for superposition and homogeneity:

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= \sin(a)(\alpha x_1 + \beta y_1) + \exp(b)(\alpha x_2 + \beta y_2) \\ &= \alpha \sin(a)x_1 + \alpha \exp(b)x_2 + \alpha \sin(a)y_1 + \alpha \exp(b)y_2 \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Hence this function is **linear**.

(d)

$$f(x_1, x_2) = x_2 - x_1 + 3$$

**Answer:** Check for superposition and homogeneity:

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= (\alpha x_2 + \beta y_2) - (\alpha x_1 + \beta y_1) + 3 \\ &\neq \alpha(x_2 - x_1 + 3) + \beta(y_2 - y_1 + 3) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

This function is **not linear**; in fact, this is an affine function (see **Note 1: Subsection 1.4.2** for more details).

#### 4. Spanning Set

**Learning Goal:** The goal of this problem is to connect Gaussian Elimination and linear (in)dependence to the concept of span. Another goal is to be comfortable with the geometric representation of span.

- (a) For what values of  $b_1, b_2, b_3$  is the following system of linear equations consistent? (“Consistent” means there is at least one solution. Please see **Note 1B: Subsection 1.2.4.2** for more details on consistency of a system.)

$$\mathbf{A}\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

**Answer:** For the system of linear equations to be consistent, there must exist some  $\vec{x}$  such that the equality above holds. Performing matrix vector multiplication, we can rewrite the above equality as

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The LHS of the above equation is a linear combination of the columns of  $\mathbf{A}$ . So, the system will be consistent as long as  $\vec{b}$  can be written as a linear combination of the columns of  $\mathbf{A}$ . In other words,  $\vec{b}$  needs to be in  $\text{range}(\mathbf{A}) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ . Please see **Note 3: Section 3.3** for the relation between linear dependence and span.

**Note:** Any system  $\mathbf{A}\vec{x} = \vec{b}$  will be consistent if and only if,  $\vec{b} \in \text{range}(\mathbf{A})$  i.e.  $\vec{b}$  belongs in the span of the columns of  $\mathbf{A}$ . This is a very key concept. Matrix multiplication can be thought of as

**remapping a vector (or a set of vectors) to a new coordinate plane. The set of vectors that can be reached in the span of the columns of the matrix transformation.**

Performing Gaussian Elimination on the augmented matrix  $[\mathbf{A}|\vec{b}]$ , we have:

$$\begin{aligned} \left[ \begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 1 & b_2 \\ 0 & 0 & b_3 \end{array} \right] &\rightarrow \left[ \begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -3 & b_2 - 2b_1 \\ 0 & 0 & b_3 \end{array} \right] \text{ using } R_2 \leftarrow R_2 - 2R_1 \\ &\rightarrow \left[ \begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & (2b_1 - b_2)/3 \\ 0 & 0 & b_3 \end{array} \right] \text{ using } R_2 \leftarrow -R_2/3 \\ &\rightarrow \left[ \begin{array}{cc|c} 1 & 0 & (2b_2 - b_1)/3 \\ 0 & 1 & (2b_1 - b_2)/3 \\ 0 & 0 & b_3 \end{array} \right] \text{ using } R_1 \leftarrow R_1 - 2R_2 \end{aligned}$$

Looking at the last row of the row reduced matrix, that  $b_3$  needs to be zero to avoid inconsistency of the system. On the other hand, solution would exist for any scalar values of  $b_1$  and  $b_2$ .

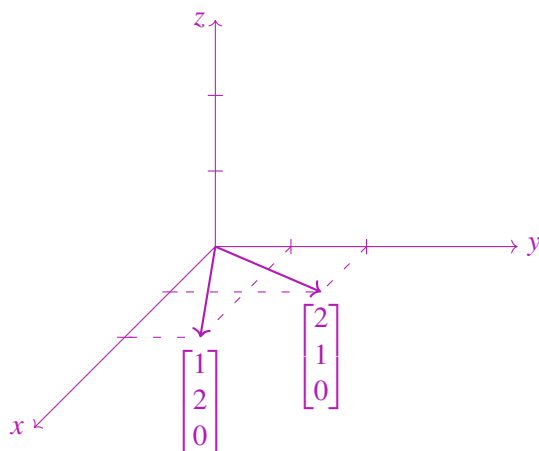
(b) What is the geometry represented by  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$  ?

**Answer:**  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$  contains any vector  $\vec{v}$  that can be written as

$$\vec{v} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$

where  $\alpha_1$  and  $\alpha_2$  are scalars. We realize that from part (a) and part (b) that any vector whose third component is 0 can be written in this form. Hence, the required span is the set of all vectors that can

be written in the form  $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$ . Geometrically, the span is the set of all vector in the  $xy$ -plane in  $\mathbb{R}^3$ .



(c) Find out if  $\vec{v}_1 = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$  is in  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ . What about  $\vec{v}_2 = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$  ?

**Answer:**

Just at first glance, we can immediately see that  $\vec{v}_2$  is not in the span since in part (a), we determined that  $b_3$  must be 0.

However, we can use Gaussian Elimination again to find out:

$$\begin{aligned} \left[ \begin{array}{cc|c} 1 & 2 & -3 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{array} \right] &\rightarrow \left[ \begin{array}{cc|c} 1 & 2 & -3 \\ 0 & -3 & 11 \\ 0 & 0 & 0 \end{array} \right] \text{ using } R_2 \leftarrow R_2 - 2R_1 \\ &\rightarrow \left[ \begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 1 & -11/3 \\ 0 & 0 & 0 \end{array} \right] \text{ using } R_2 \leftarrow -R_2/3 \\ &\rightarrow \left[ \begin{array}{cc|c} 1 & 0 & 13/3 \\ 0 & 1 & -11/3 \\ 0 & 0 & 0 \end{array} \right] \text{ using } R_1 \leftarrow R_1 - 2R_2 \end{aligned}$$

So we can write that  $x_1 = \frac{13}{3}$  and  $x_2 = -\frac{11}{3}$ .

$$\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} = \frac{13}{3} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{11}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

So  $\vec{v}_1$  is in the span.

If we follow the same process for  $\vec{v}_2$ , we are going to see that the system is inconsistent, i.e. it cannot be solved for  $x_1$  and  $x_2$ , which means  $\vec{v}_2$  cannot be in the span.

- (d) Reflect on your answer from part(b) and find out  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \right\}$ .

**Answer:** From part (b), we found that  $\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$  is in  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ , i.e.  $\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$  is linearly dependent on  $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ . So any vector  $\vec{v} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \right\}$  can be written as:

$$\begin{aligned} \vec{v} &= \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \\ &= \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \frac{13}{3} \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{11}{3} \alpha_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \left( \alpha_1 + \frac{13}{3} \alpha_3 \right) \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \left( \alpha_2 - \frac{11}{3} \alpha_3 \right) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Since  $(\alpha_1 + \frac{13}{3}\alpha_3)$  and  $(\alpha_2 - \frac{11}{3}\alpha_3)$  are both scalars in the above equation, we can say that  $\vec{v} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ . Similarly you can show that any vector  $\vec{u} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$  would also belong in  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \right\}$ . So  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(e) What is a possible choice for  $\vec{v}$  that would make  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3$  ?

**Answer:** From part (a), we realize that any vector whose last component is 0 can be written as a linear combination of the two vectors already in the set. Hence, if we include, for example,  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  into the set, then we should be able to reach any vector in  $\mathbb{R}^3$ . Any vector whose last component is non-zero is a valid addition to the set to achieve the desired span.

**Note:** We need *at least*  $n$  linearly independent vectors  $\in \mathbb{R}^n$  to span the entirety of  $\mathbb{R}^n$ . In other words,  $\text{span} \{ \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \} = \mathbb{R}^n$ , for a linearly independent set of  $\vec{a}_1, \dots, \vec{a}_n$ .