CSM 16A Designing Information Devices and Systems I
Spring 2021 Week 3

1. A Tale of Two Spaces

Learning Goal: The goal of this problem is to understand subspaces, basis vectors, and dimension. Please look into **Note 8 Section 8.1** for more on subspaces.

(a) Consider the set U, which is a subset of $\mathbb{R}^3$, defined below. Is U a subspace?

$$U = \left\{ \begin{bmatrix} x \\ 0 \\ x+y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

(b) Find a basis for U. What is its dimension?

(c) Consider the set V, which is a subset of $\mathbb{R}^3$, defined below. Is V a subspace?

$$V = \left\{ \begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix} \mid z \in \mathbb{R} \right\}$$

(d) Find a basis for V. What is its dimension?

(e) Can you express the basis vector(s) you found in part (d) as a linear combination of the basis vector(s) you found in part (b)? Why or why not?

2. Nullspace and Loss of Dimensionality [WALK-THROUGH]

Learning Goal: The goal of this problem to understand the relationship between nullspace and loss of dimensionality/ invertibilty.

Please look into **Note 8 Section 8.3** to learn how the dimension of the output space depends on the nullspace.

Answer the following questions for all three parts:

- Find the columnspace and nullspace of the following matrices in terms of basis vectors.
- What are the dimensions of the columnspace/nullspace? Remember that the Rank Nullity theorem shows that the number of columns of a matrix A = dim(N(A)) [nullity of matrix A] + dim(C(A)) [rank of matrix A]
- What kind of geometry is represented by the columnspace/nullspace?
- Is the matrix invertible?

(a) Consider a matrix **P**:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Consider a matrix $\mathbf{Q}$:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Consider a matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Fundamental Subspaces [WALK-THROUGH]

Learning Goal: The goal of this problem to practice finding the columnspace and nullspace of a matrix.

Please look into **Note 8 Section 8.2-8.4** to learn about the significance of columnspace and nullspace.

Consider a matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -2 & -1 \end{bmatrix}$$

(a) Find a basis for the column space of $\mathbf{A}$. What is the dimension of this space?

(b) Find a basis for the nullspace of $\mathbf{A}$. What is the dimension of this space?

4. Proof on Nullspace

Learning Goal: The goal of this problem is to practice some more proof development skills.

(a) **Show that if a square matrix A is invertible, then it has a trivial nullspace.**
Please look into **Note 8 Section 8.3** to learn how the dimension of the output space depends on the nullspace.

5. Non-invertible Square Matrix

Learning Goal: The goal of this problem is to understand loss of dimensionality in relation to nullspace.

(a) For given matrices $\mathbf{A} \in \mathbb{R}^{3\times 2}$ and $\mathbf{B} \in \mathbb{R}^{2\times 3}$, the products will be square matrices: $\mathbf{AB} \in \mathbb{R}^{3\times 3}$ and $\mathbf{BA} \in \mathbb{R}^{2\times 2}$. Show that $\mathbf{AB}$ is not invertible.
Please look into **Note 8 Section 8.3** to learn how the dimension of the output space depends on the nullspace.
Hint: *A good proof strategy is to utilize what we have already proven before. Is there a way we can use the result in Question 4, "Proof on Nullspace"?*