EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 1A

1. Gaussian Elimination

Use Gaussian elimination to solve the following systems. Does a solution exist? Is it unique?

(a)

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 8 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Answer:
No solution. Performing Gaussian Elimination on the augmented matrix will lead to a row of zeros with a non zero constant term.

(c)

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

Answer:
There are an infinite number of solutions. One solutions is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

(d) True or False: A system of equations with more equations than unknowns will always have either infinite solutions or no solutions.

Answer: False, a counter example of this is when we have N equations and K unknowns ($N > K$) and $N-K$ of the equations are linear combinations of the first K. This means that there are actually K unique equations and K unique unknowns; therefore a unqiue solution will exist. This can be observed when Gaussian elimination is performed and the last $N-K$ rows are all 0.

For example, here is a system of four equations and two unknowns,

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 4 & 2 \\ 2 & 5 & 3 \end{array}\right] \xRightarrow{-2R_1+R_3 \to R_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 5 & 3 \end{array}\right] \xRightarrow{-2R_1-R_2+R_3 \to R_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{1}$$

Solving, we get a single exact solution, $x = -1$ and $y = 1$.

(e) **(Practice)**

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Answer:

There are an Infinite number of solutions. One solutions is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ \frac{1}{2} \end{bmatrix}$$

Performing Gaussian elimination will lead to a row of zeros on the bottom row.

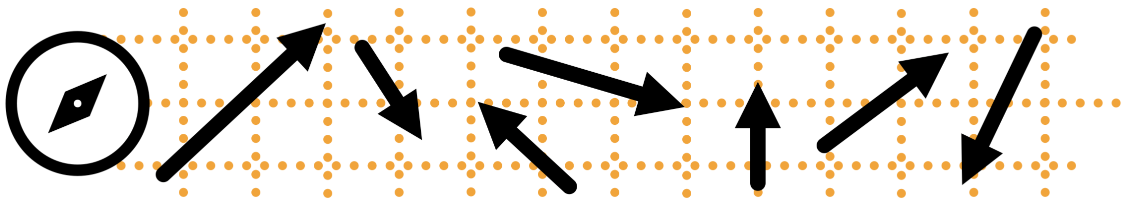
(f) **(Practice)**

$$\begin{bmatrix} 2x & + & 4y & + & 2z & = & 8 \\ x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & 4 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

2. Vectors



A vector is an ordered list of numbers. For instance, a point on a plane (x,y) is a vector! We label vectors using an arrow overhead $\vec{v}$, and since vectors can live in ANY dimension of space we'll need to leave our notation general $(x,y) \rightarrow \vec{v} = (v_1, v_2, \ldots)$. Below are few more examples (the left-most form is the general definition):

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \vec{a} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3 \qquad \vec{b} = \begin{bmatrix} 2.4 \\ 5.3 \end{bmatrix} \in \mathbb{R}^2$$

Just to unpack this a bit more, $\vec{b} \in \mathbb{R}^3$ in english means "vector $\vec{b}$ lives in 3-Dimensional space".

- The $\in$ symbol literally means "in"
- The $\mathbb{R}$ stands for "real numbers" (FUN FACT: $\mathbb{Z}$ means "integers" like $-2, 4, 0, \ldots$)
- The exponent $\mathbb{R}^{n \leftarrow}$ indicates the dimension of space, or the amount of numbers in the vector.

One last thing: it is standard to write vectors in column-form, like seen with $\vec{a}, \vec{b}, \vec{x}$ above. We call these *column vectors*, in contrast to horizontally written vectors which we call *row vectors*.

Okay, let's dig into a few examples:

(a) Which of the following vectors live in $\mathbb{R}^2$ space?

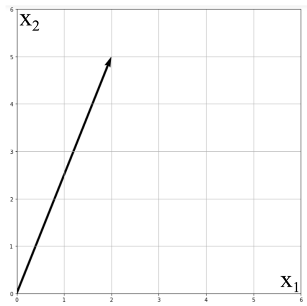
$$i.\ \begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad ii.\ \begin{bmatrix} 5 \\ 0 \\ 3 \\ 5 \end{bmatrix} \qquad iii.\ \begin{bmatrix} -4.76 \\ 1.32 \\ 0.01 \end{bmatrix} \qquad iv.\ \begin{bmatrix} -20 \\ 100 \end{bmatrix}$$

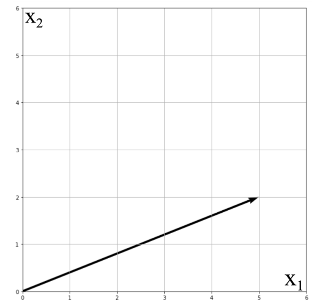
Solution/Answer:
i. Yes ii. No iii. No iv. Yes
Remember $\mathbb{R}^2$ means 2D space, which hosts vectors with 2 terms.
We count and see only i. and iv. have 2 terms.

(b) Graphically show the vectors (either in a sketch with axes, or a plot on a computer):

$$i.\ \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad ii.\ \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Solution/Answer: Although these vectors look similar, remember that the ordering matters!

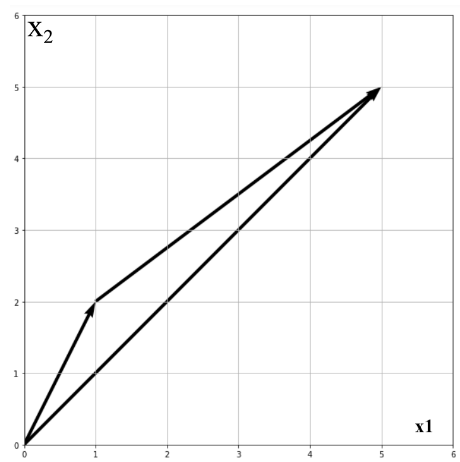




(c) Compute the sum $\vec{a}+\vec{b}=\vec{c}$ from the vectors below, and then graphically sketch or plot these vectors. (show them in a way that forms a triangle; also is there only one possible triangle?)

$$\vec{a}=\begin{bmatrix}1\\2\end{bmatrix} \qquad \vec{b}=\begin{bmatrix}4\\3\end{bmatrix}$$

Solution/Answer:



Computation is done element-wise:

$$\vec{c} = \vec{a}+\vec{b} = \begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}$$