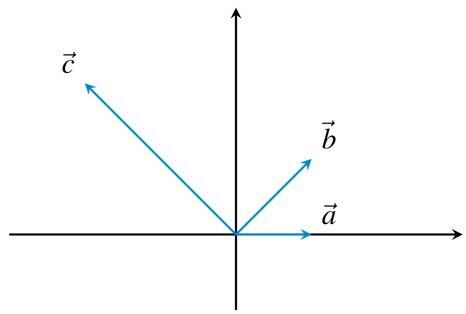
EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 2A

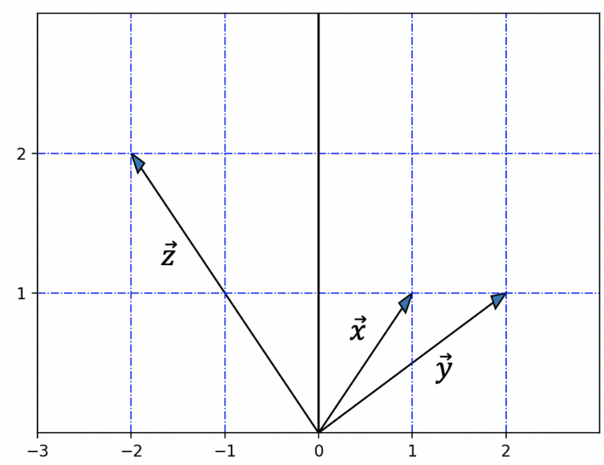
1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y}=\begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z}=\begin{bmatrix}-2\\2\end{bmatrix}$. Draw these vectors on a sheet of paper.

Answer:



(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$ so that we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

Answer:

$$\alpha\vec{x}+\beta\vec{y}=\vec{z}$$

$$\alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$$

$$\begin{cases} \alpha + \beta \cdot 2 = -2 \\ \alpha + \beta = 2 \end{cases}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(c) Solve for α, β.

Answer: We start by writing the system in the augmented matrix form

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 1 & 1 & 2 \end{array}\right]$$

Then we solve the system using Gaussian Elimination. First, we subtract the second row by the first row:

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -1 & 4 \end{array}\right]$$

Next, we multiply the second row by -1 to solve for β.

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -4 \end{array}\right]$$

We get $\beta = -4$. Then, we take the first row and subtract it by the second row*2.

$$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -4 \end{array}\right]$$

So the solution is $\alpha = 6$ and $\beta = -4$.

2. Span basics

(a) What is span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer: span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ contains any vector $\vec{v}$ that can be written as

$$\vec{v} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

We realize that any vector whose last component is 0 can be written in this form and any vector whose last component is nonzero cannot. Hence, the required span is the set of all vectors that can be written in the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$.

(b) Is $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer: Yes. We realize from inspection that

$$\begin{bmatrix}5\\5\\0\end{bmatrix} = \frac{5}{3}\begin{bmatrix}1\\2\\0\end{bmatrix} + \frac{5}{3}\begin{bmatrix}2\\1\\0\end{bmatrix}$$

(c) What is a possible choice for $\vec{v}$ that would make $\text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \vec{v}\right\} = \mathbb{R}^3$?

Answer: From part (a), we realize that any vector whose last component is 0 can be written as a linear combination of the two vectors already in the set. Hence, if we include, for example, $\begin{bmatrix}0\\0\\1\end{bmatrix}$ into the set, then we should be able to reach any vector in $\mathbb{R}^3$. Any vector whose last component is non-zero is a valid addition to the set to achieve the desired span.

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? ("Consistent" means there is at least one solution.)

$$\begin{bmatrix}1 & 2\\2 & 1\\0 & 0\end{bmatrix}\vec{x} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

Answer: For the system of linear equations to be consistent, there must exist some x such that the equality above holds. Performing matrix vector multiplication, we can rewrite the above equality as

$$x_1\begin{bmatrix}1\\2\\0\end{bmatrix} + x_2\begin{bmatrix}2\\1\\0\end{bmatrix} = \vec{b}$$

The question now becomes: which vectors $\vec{b}$ can be written in the above form i.e as a linear combination of the columns of A? This is exactly the definition of span, and the answer must be the same as that from part (a).

All Rights Reserved. This may not be publicly shared without explicit permission.

3. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

(a)

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

Answer: Suppose we have some arbitrary $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. With some scalars a_i we can express $\vec{q}$:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \left(\frac{a_1}{\alpha}\right)\alpha\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n.$$

Since $\vec{q}$ is also expressible as a linear combination of $\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ we have shown that $\vec{q} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. With some scalars b_i we can express $\vec{r}$:

$$\vec{r} = b_1(\alpha\vec{v}_1) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = (b_1\alpha)\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Since $\vec{r}$ is expressible as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ we have shown that, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we now have $\text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.

(b) **(Practice)**

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

Answer: Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n$$

The latter equality comes from adding and subtracting $a_1\vec{v}_2$ and combining like terms. Thus, we have shown that $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.