EECS 16A Designing Information Devices and Systems I
Spring 2022 Discussion 4A

1. Mechanical Inverses

For each sub-part below, determine whether or not the inverse of $\mathbf{A}$ exists.
If it exists, compute the inverse using the Gauss-Jordan method.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer: We use Gaussian elimination (also known as the Gauss-Jordan method):

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{9}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$.

(b) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Answer: We can again use the Gauss-Jordan method:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{a}R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - cR_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{c}{a}b & -\frac{c}{a} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{d-\frac{c}{a}b}R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-\frac{c}{a}}{d-\frac{c}{a}b} & \frac{1}{d-\frac{c}{a}b} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{b}{a}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{b}{a}\frac{c}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

Therefore, we get that $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
This is a known formula which, if you find useful, you can use for any general 2x2 matrix.
Note that the matrix does not have an inverse if $ad - bc = 0$.

(c) $\mathbf{A} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & -2 & 4 \end{bmatrix}$

Answer:
Since have a non-square matrix $\mathbf{A}$, there cannot be a unique inverse.
We can understand this from the fact that for $\vec{y} = \mathbf{A}\vec{x}$ the vectors $\vec{x} \in \mathbb{R}^3$ and $\vec{y} \in \mathbb{R}^2$ live in different spaces. This leads us to conclude that there cannot be a unique $\vec{x}$ for each $\vec{y}$.

(d) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

Answer: We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

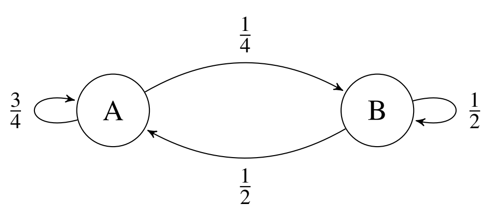
$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array}\right].$$

While row-reducing, we notice that the second column doesn't have a pivot (and that there is also a row of zeros). Therefore, no inverse exists.

2. Transition Matrix

Suppose we have a network of pumps as shown in the diagram below. Let us describe the state of A and B using a state vector $\vec{x}[n] = \begin{bmatrix} x_A[n] \\ x_B[n] \end{bmatrix}$ where $x_A[n]$ and $x_B[n]$ are the states at time-step n.



(a) Find the state transition matrix S, such that $\vec{x}[n+1] = S\,\vec{x}[n]$.
Separately find the sum of the terms for each column vector in S. Do you notice any pattern?

Answer:

We can write the following equations by examining the state transition diagram:

$$\begin{aligned} x_A[n+1] &= (3/4)\,x_A[n] + (1/2)\,x_B[n] \\ x_B[n+1] &= (1/4)\,x_A[n] + (1/2)\,x_B[n] \end{aligned}$$

From here we can directly write down the state transition matrix as:

$$S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

Note the columns of S each sum to 1, which ensures we have a physical system satisfying conservation.

(b) Let us now find the matrix S^{-1} such that we can recover the previous state $\vec{x}[n-1]$ from $\vec{x}[n]$.
Specifically, solve for S^{-1} such that $\vec{x}[n-1] = S^{-1}\,\vec{x}[n]$.

Answer: We can use Gauss-Jordan method to solve for matrix S^{-1}:

$$\left[\begin{array}{cc|cc} 3/4 & 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{4}{3}R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow -4R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right] \xrightarrow{R_2 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \xrightarrow{R_2 \leftarrow -\frac{3}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 3 \end{array}\right].$$

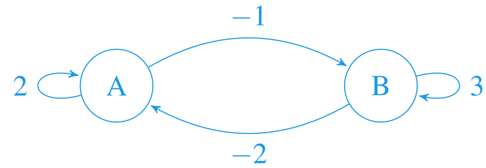
Therefore:

$$S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

Note that the columns of S^{-1} still sum to 1 despite matrix elements lying outside $[0,1]$.
So while this is not physical, the inverse process obeys conservation as expected.

(c) Now draw the state transition diagram that corresponds to the S^{-1} that you just found.
Also find the sum of the terms for each column vector in S^{-1}. Do you notice any pattern?

Answer:



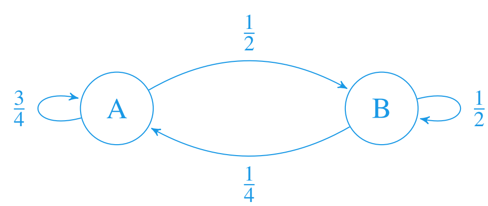
We can write the following equations from the state transition diagram:

$$x_A[n-1] = 2\,x_A[n] - 2x_B[n]$$
$$x_B[n-1] = -x_A[n] + 3\,x_B[n]$$

The matrix S^{-1} can be thought of as the transition that *turns back time* for the pump system. **Although it is non-physical**, the weights that have an absolute value greater than 1 can be thought of as "generating" water, and the weights that have negative weight can be thought of as "destroying" water. However, note (as seen above) that the outflow weights of each node still sum to 1 (i.e. the columns of S^{-1} still sum to 1). This means that in total all of the water is being conserved during the transition between time steps, even when time is reversed.

(d) Redraw the diagram from the first part of the problem, but now with the directions of the arrows reversed. Let us call the state transmission matrix of this "reversed" state transition diagram T. Does $T = S^{-1}$?

Answer:



After drawing the "reversed" state transition diagram, we can write the following equations:

$$x_A[n+1] = (3/4)\,x_A[n] + (1/4)\,x_B[n]$$
$$x_B[n+1] = (1/2)\,x_A[n] + (1/2)\,x_B[n]$$

From here, we can directly write down the state transition matrix as:

$$T = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Note that $T \neq S^{-1}$. What we have actually found is that T is equal to the *transpose* of S, denoted by $S^{\top}$ (the superscript $\top$ denotes the transpose of a matrix). The transpose of a matrix is when its rows become its columns. In general, a matrix's inverse and its transpose are not equal to each other.

(e) Suppose we start in the state $\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$. Compute the state vector after two time-steps, $\vec{x}[3]$.

Answer: There are two ways to approach this problem:

i. Compute states successively: $\vec{x}[2] = S\,\vec{x}[1]$, then $\vec{x}[3] = S\,\vec{x}[2]$

ii. Compute directly: $\vec{x}[3] = S\,S\,\vec{x}[1]$

They are equivalent because of the fact that matrix multiplication is associative:

$$\vec{x}[3] = S(S\,\vec{x}[1]) = (S\,S)\,\vec{x}[1]$$

We start with method i.

$$\vec{x}[2] = S\,\vec{x}[1] = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \cdot 12/4 + 12/2 \\ 1 \cdot 12/4 + 12/2 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$$

$$\vec{x}[3] = S\,\vec{x}[2] = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \cdot 15/4 + 9/2 \\ 1 \cdot 15/4 + 9/2 \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix} \qquad \square$$

Alternatively we can use method ii.

$$\vec{x}[3] = S\,S\,\vec{x}[1] \quad \rightarrow \quad S^2 = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 9/16 + 1/8 & 3/8 + 1/4 \\ 3/16 + 1/8 & 1/8 + 1/4 \end{bmatrix} = \begin{bmatrix} 11/16 & 5/8 \\ 5/16 & 3/8 \end{bmatrix}$$

$$\vec{x}[3] = S^2\,\vec{x}[1] = \begin{bmatrix} 11/16 & 5/8 \\ 5/16 & 3/8 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 11 \cdot 3/4 + 5 \cdot 6/4 \\ 5 \cdot 3/4 + 3 \cdot 6/4 \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix} \qquad \square$$

