EECS 16A Designing Information Devices and Systems I
Spring 2022 Discussion 5B

1. Mechanical Determinants

(a) Compute the determinant of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Answer:

We can use the form of a 2×2 determinant from lecture:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Therefore,

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 0 \cdot 0 = 6$$

(b) Compute the determinant of $\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$.

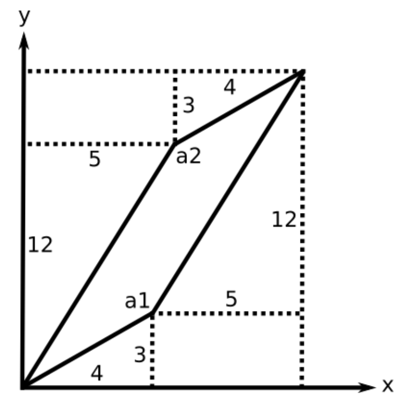
Answer:

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \cdot \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

Therefore,

$$\begin{aligned} \det\left(\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}\right) &= 2 \cdot \det\left(\begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix}\right) + 3 \cdot \det\left(\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}\right) \\ &= 2 \cdot (0 - (-4)) + 3 \cdot (10 - (-1)) + 1 \cdot (8 - 0)) \\ &= 8 + 33 + 8 \\ &= 49 \end{aligned}$$

(c) We know that the determinant of a matrix represents the multi-dimensional volume formed by the column vectors. Explain geometrically why the determinant of a matrix with linearly dependent column vectors is always 0.



Answer: Consider an example in $\mathbb{R}^2$. If we have two vectors that are linearly independent, we can form a parallelogram with them and calculate its area, which will be a nonzero value. Thus, we'll have a nonzero determinant as well. However, if the vectors are linearly dependent, we only have one dimension (since the other dimension would have been compressed to zero), so our "parallelogram" would have an area of 0, corresponding to a determinant with a value of 0.
This idea generalizes to N dimensions. If we have fewer than N linearly independent vectors, then the multi-dimensional volume will have at least 1 dimension compressed to 0, giving us 0 volume and 0 determinant.

2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and the associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$
Do you observe anything?
Answer:
Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.
For the eigenvalue $\lambda = 1$:

$$(\mathbf{M} - 1\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}\vec{x} = \vec{0}$$

From the second equation in the system, $x_2 = 0$, with any solution having the form $\begin{bmatrix} 1 \\ 0 \end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is thus $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 9$:

$$\begin{aligned}
(\mathbf{M} - 9\mathbf{I})\vec{x} &= \vec{0} \\
\left(\begin{bmatrix}1 & 0 \\ 0 & 9\end{bmatrix} - 9\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\right)\vec{x} &= \vec{0} \\
\left(\begin{bmatrix}1 & 0 \\ 0 & 9\end{bmatrix} - \begin{bmatrix}9 & 0 \\ 0 & 9\end{bmatrix}\right)\vec{x} &= \vec{0} \\
\begin{bmatrix}-8 & 0 \\ 0 & 0\end{bmatrix}\vec{x} &= \vec{0}
\end{aligned}$$

From the first equation in the system, $x_1 = 0$, so any solution must take the form $\begin{bmatrix}0 \\ 1\end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is span $\left\{\begin{bmatrix}0 \\ 1\end{bmatrix}\right\}$.

We observe that the eigenvalues are just the diagonal entries. Since the matrix is diagonal, multiplying the diagonal matrix $\mathbf{D}$ with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries d_i of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(b) $\mathbf{M} = \begin{bmatrix}0 & 1 \\ -2 & -3\end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix}0-\lambda & 1 \\ -2 & -3-\lambda\end{bmatrix}\right) = 0$$

$$\begin{aligned}
-\lambda(-3-\lambda) + 2 &= 0 \\
\lambda^2 + 3\lambda + 2 &= 0 \\
(\lambda + 2)(\lambda + 1) &= 0 \\
\lambda &= -1, -2
\end{aligned}$$

$\lambda = -1$:

$$\left[\begin{array}{cc|c} 0-(-1) & 1 & 0 \\ -2 & -3-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}-t \\ t\end{bmatrix} = \begin{bmatrix}-1 \\ 1\end{bmatrix} t$$

The eigenspace for $\lambda = -1$ is span$\left\{\begin{bmatrix}-1 \\ 1\end{bmatrix}\right\}$.

$\lambda = -2$:

$$\left[\begin{array}{cc|c} 0-(-2) & 1 & 0 \\ -2 & -3-(-2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2/2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}-t/2 \\ t\end{bmatrix} = \begin{bmatrix}-1/2 \\ 1\end{bmatrix} t$$

The eigenspace for $\lambda = -2$ is span$\left\{\begin{bmatrix}-1/2 \\ 1\end{bmatrix}\right\}$.

(c) Without calculation, determine whether the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$. What are the corresponding eigenvectors?

Answer: Multiplying the identity matrix with any vector in $\mathbb{R}^n$ produces the same vector, that is, $\mathbf{I}\vec{x} = \vec{x} = 1 \cdot \vec{x}$. Therefore, $\lambda = 1$. Since $\vec{x}$ can be any vector in $\mathbb{R}^n$, the corresponding eigenvectors are all vectors in $\mathbb{R}^n$.

3. Steady and Unsteady States

You're given the matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$.

(a) The eigenvalues of $\mathbf{M}$ are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = \frac{1}{2}$. Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors corresponding to the eigenvalues. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

Answer:

$$\begin{aligned} \mathbf{M}^n\vec{x} &= \mathbf{M}^n(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3) \\ &= \alpha\mathbf{M}^n\vec{v}_1 + \beta\mathbf{M}^n\vec{v}_2 + \gamma\mathbf{M}^n\vec{v}_3 \\ &= 1^n\alpha\vec{v}_1 + 2^n\beta\vec{v}_2 + \left(\frac{1}{2}\right)^n\gamma\vec{v}_3 \end{aligned}$$

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	-
0	$\neq 0$	$\neq 0$	No	-
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	-
$\neq 0$	$\neq 0$	$\neq 0$	No	-

(b) (**Practice**) Find the eigenspaces associated with the eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$
ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$
iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Answer:

i. $\lambda = 1$:

$$\left[\begin{array}{c|c} \mathbf{M}-\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{G.E.}{\to} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

This means that

$$span\{\vec{v}_1\} = span\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

ii. $\lambda = 2$:

$$\left[\begin{array}{c|c} \mathbf{M}-2\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} \frac{-3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{G.E.}{\to} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$$

This means that

$$span\{\vec{v}_2\} = span\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}$$

iii. $\lambda = \frac{1}{2}$:

$$\left[\begin{array}{c|c} \mathbf{M}-\frac{1}{2}\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array}\right] \overset{G.E.}{\to} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$$

This means that

$$span\{\vec{v}_3\} = span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$