EECS 16A Designing Information Devices and Systems I
Spring 2022 Discussion 6A

1. True or False?

For each of the following subparts below, prove whether the statement is True or False.

(a) There exists an invertible $n \times n$ matrix A for which $A^2 = 0$.

Answer: False

Let's left multiply and right multiply A^2 by A^{-1} so we have $A^{-1}AAA^{-1}$. By associativity of matrix multiplication, we have $(A^{-1}A)(AA^{-1}) = I_n I_n = I_n$ where I is the identity matrix. However, if A^2 were 0, then $(A^{-1}A)(AA^{-1}) = A^{-1}A^2A^{-1} = 0$ where 0 is a matrix of all zeros, hence resulting in a contradiction.

(b) If A is an invertible $n \times n$ matrix, then for all vectors $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ has a unique solution.

Answer: True

If A is invertible, then there is a unique matrix A^{-1}. Left multiply the equation by A^{-1}, and we will have $A^{-1}A\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}$, where $\vec{x}$ is a unique vector.

(c) If A and B are invertible $n \times n$ matrices, then the product AB is invertible.

Answer: True

$(AB)^{-1} = B^{-1}A^{-1}$.
Note that $ABB^{-1}A^{-1} = I$ and $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$

(d) The two vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ form a basis for the subspace, span$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$.

Answer: True.

Span$(\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\})$ spans the x-y plane in $\mathbb{R}^3$. Since $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ are linearly independent, they form a basis for the x-y plane in $\mathbb{R}^3$ as well.

(e) The dimension of the subspace, span$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$, is 3.

Answer: False.

Since the basis of the subspace, span$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$, has two vectors, the dimension of the subspace is 2.

(f) A set of n linearly dependent vectors in $\mathbb{R}^n$ can span $\mathbb{R}^n$.

Answer: False

A set of n linearly dependent vectors span some subspace of dimension $0 < \dim(A) < n$ in $\mathbb{R}^n$.
Note: It is incorrect to say the set of linearly dependent vectors spans $\mathbb{R}^{n-1}$ for two reasons. First, you don't know what the dimension is of the subspace it spans, which could be less than $n-1$. Second, there is no such thing as $\mathbb{R}^{n-1} \in \mathbb{R}^n$. The vectors are "in" $\mathbb{R}^n$ based on how many elements are in the vector, and a set of vectors spans some subspace (potentially the entire space).

2. Are eigenvectors linearly independent?

Suppose we have a square matrix $\mathbf{A}^{n\times n}$ with 'n' distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ (meaning that $\lambda_i \neq \lambda_j$ when $i \neq j$) and 'n' corresponding eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Prove that any two eigenvectors $\vec{v}_i, \vec{v}_j$ (for $i \neq j$) are linearly independent.

HINT: Begin proof by contradiction: Suppose that $\vec{v}_i$ and $\vec{v}_j$ correspond to distinct eigenvalues, so that $(\lambda_i - \lambda_j) \neq 0$, and are linearly dependent. Show this leads to a nonsensical equality after applying $\mathbf{A}$*. (Please ask the staff for guidance if this hint is too vague!)*

Answer:
PROOF BY CONTRADICTION:
Suppose $\vec{v}_i$ and $\vec{v}_j$ correspond to distinct eigenvalues such that $(\lambda_i - \lambda_j) \neq 0$ and are linearly dependent, meaning $\alpha\vec{v}_i + \beta\vec{v}_j = \vec{0}$.

NOTE: We know that both $\alpha \neq 0$ and $\beta \neq 0$ since any zero constant would imply that one of the eigenvectors is $\vec{0}$, which by definition of an eigenvector cannot be true.

Let $\vec{u} = \alpha\vec{v}_i + \beta\vec{v}_j = \vec{0}$. By definition $\mathbf{A}\,\vec{u} = \mathbf{A}\,\vec{0} = \vec{0}$.
However ...

$$\begin{aligned}
\mathbf{A}\,\vec{u} &= \mathbf{A}(\alpha\vec{v}_i + \beta\vec{v}_j) = \alpha\,\mathbf{A}\,\vec{v}_i + \beta\,\mathbf{A}\,\vec{v}_j \\
&= \alpha\,\lambda_i\,\vec{v}_i + \beta\,\lambda_j\,\vec{v}_j \\
&= \lambda_i(\alpha\vec{v}_i + \beta\vec{v}_j) + (\lambda_j - \lambda_i)\,\beta\,\vec{v}_j \\
&= \lambda_i\,\vec{u} + (\lambda_j - \lambda_i)\,\beta\,\vec{v}_j \\
&= (\lambda_j - \lambda_i)\,\beta\,\vec{v}_j \qquad = \vec{0}
\end{aligned}$$

Since all three components $(\lambda_j - \lambda_i)$, β, and $\vec{v}_j$ cannot be zero by construction (and/or definition), we've arrived at a contradiction suggesting that the eigenvectors $\vec{v}_i$ and $\vec{v}_j$ MUST be linearly independent! □

3. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

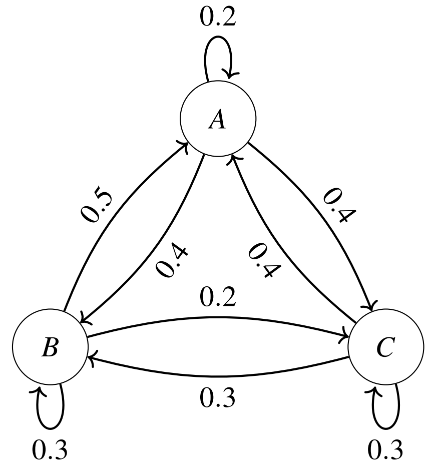

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix $\mathbf{T}$ representing the pumps system.

Answer:

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of $\mathbf{T}$. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

Answer:

We know $\lambda_1 = 1$ is the eigenvalue corresponding to the steady state eigenvector. Therefore,

$$\begin{aligned} T\vec{x} &= 1\vec{x} \\ &= \lambda_1 \vec{x} \\ \Rightarrow \vec{x} &\in N(\mathbf{T} - 1 \cdot \mathbf{I}) \\ \vec{x} &\in N\left(\begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \\ \vec{x} &\in N\left(\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix} \right). \end{aligned}$$

In order to row reduce $\mathbf{T} - 1 \cdot \mathbf{I}$ we use Gaussian elimination. We also convert to fractions:

$$\begin{bmatrix} -\frac{4}{5} & \frac{1}{2} & \frac{2}{5} \\ \frac{2}{5} & -\frac{7}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{1}{5} & -\frac{7}{10} \end{bmatrix} \overset{R_1 \leftarrow -5/4 R_1}{\Longrightarrow} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ \frac{2}{5} & -\frac{7}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{1}{5} & -\frac{7}{10} \end{bmatrix} \underset{R_3 \leftarrow R_3 - 2/5 R_1}{\overset{R_2 \leftarrow R_2 - 2/5 R_1}{\Longrightarrow}} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ 0 & -\frac{9}{20} & \frac{1}{2} \\ 0 & \frac{9}{20} & -\frac{1}{2} \end{bmatrix}$$

$$\overset{R_2 \leftarrow -20/9 R_2}{\Longrightarrow} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ 0 & 1 & -\frac{10}{9} \\ 0 & \frac{9}{20} & -\frac{1}{2} \end{bmatrix} \overset{R_3 \leftarrow R_3 - 9/20 R_2}{\Longrightarrow} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ 0 & 1 & -\frac{10}{9} \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 \leftarrow R_1 + 5/8 R_2}{\Longrightarrow} \begin{bmatrix} 1 & 0 & -\frac{43}{36} \\ 0 & 1 & -\frac{10}{9} \\ 0 & 0 & 0 \end{bmatrix}$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a vector describing the steady state, then we can set x_3 to be the free variable. Thus we can write the form any steady state vector should take using the first two equations represented by the row reduced matrix:

$$\begin{aligned} x_1 - \tfrac{43}{36} x_3 &= 0 \\ x_2 - \tfrac{10}{9} x_3 &= 0 \\ x_3 &= \alpha \in \mathbb{R} \end{aligned} \implies \vec{x} = \begin{bmatrix} \frac{43}{36} \\ \frac{10}{9} \\ 1 \end{bmatrix} \alpha$$

(c) What does the magnitude of the other two eigenvalues λ_2 and λ_3 say about the steady state behavior of their associated eigenvectors?

Answer: The magnitude of both eigenvalues is less than 1, so in steady state, the components associated with those eigenvectors $\vec{v}_2$ and $\vec{v}_3$ will trend toward $\vec{0}$. Additionally, since $\lambda_2 < 0$, its associated eigenvector will oscillate / flip signs back and forth.

(d) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

Answer:

From the previous sub-parts we know the steady-state solution should have the form (rescaled for convenience) $\vec{x}_{ss} = \alpha \begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix}$ for any α.

But after inspecting the transition matrix we recognize that the columns each sum to one, thus, we have a conservative system, meaning that the total volume across all three reservoirs $(A_0 + B_0 + C_0)$ must remain constant at all iterations. This gives us a sufficient condition to identify α.

So far the sum, with $\alpha = 1$ of $\vec{x}_{ss}$ is $43 + 40 + 36 = 119$ (kiloliters), while the initial state starts with $A_0 + B_0 + C_0 = 129 + 109 + 0 = 238$ kiloliters. By inspection we see that $\alpha = 2$ is the proper rescaling of the steady-state eigenvector to satisfy this condition. Thus

$$\vec{x}_{ss} = \begin{bmatrix} 86 \\ 80 \\ 72 \end{bmatrix}. \qquad \square$$