# EECS 16A Designing Information Devices and Systems I
# Spring 2022 Discussion 13B

## 1. Orthonormal Matrices and Projections

An orthonormal matrix, $\mathbf{A}$, is a matrix whose columns, $\vec{a}_i$, are:

- Orthogonal (ie. $\langle \vec{a}_i, \vec{a}_j \rangle = 0$ when $i \neq j$)
- Normalized (ie. vectors with length equal to 1, $\|\vec{a}_i\| = 1$). This implies that $\|\vec{a}_i\|^2 = \langle \vec{a}_i, \vec{a}_i \rangle = 1$.

**Solution:** Note to TAs: Short matices (more columns than rows) cannot be orthonormal.

(a) Suppose that the matrix $\mathbf{A} \in \mathbb{R}^{N\times M}$ has linearly independent columns. The vector $\vec{y}$ in $\mathbb{R}^N$ is not in the subspace spanned by the columns of $\mathbf{A}$. What is the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{A}$?

**Solution:**

Remind (discussed in lecture) the class what happens when the columns of $\mathbf{A}$ are not linearly independent. When $\mathbf{A}$ has columns that are linearly dependent, then it will have a non-trivial null space. This causes $\mathbf{A}^T\mathbf{A}$ to have a non-trivial null space and therefore $(\mathbf{A}^T\mathbf{A})^{-1}$ does not exist.

**Answer:** When finding a projection onto a subspace, we're trying to find the "closest" vector in that subspace. This can be found by first finding $\vec{x}$ that minimizes $\|\vec{y} - \mathbf{A}\vec{x}\|$. From least squares, we know that $\vec{\hat{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$. The projection of $\vec{y}$ onto the columns of $\mathbf{A}$ is then $\vec{\hat{y}} = \mathbf{A}\vec{\hat{x}} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$.

(b) Show if $\mathbf{A} \in \mathbb{R}^{N\times N}$ is an orthonormal matrix then the columns, $\vec{a}_i$, form a basis for $\mathbb{R}^N$.

**Answer:**

We want to show that the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^N$. To show that the columns form a basis for $\mathbb{R}^N$ we need to show two things:

- The columns must form a set of $N$ linearly independent vectors.
- Any vector $\vec{x} \in \mathbb{R}^N$ can be represented as a linear combination of the vectors in the set.

We already know we have $N$ vectors, so first we will show they are linearly independent. We shall do this by showing that $\mathbf{A}\vec{\beta} = \vec{0}$ implies that $\vec{\beta}$ can be only $\vec{0}$.

$$\mathbf{A}\vec{\beta} = \vec{0} \tag{1}$$

$$\beta_1\vec{a}_1 + \ldots + \beta_N\vec{a}_N = \vec{0} \tag{2}$$

Then to exploit the properties of orthogonal vectors, we consider taking the inner product of each side of the above equation with $\vec{a}_i$.

$$\langle \vec{a}_i, \beta_1\vec{a}_1 + \ldots + \beta_N\vec{a}_N \rangle = \langle \vec{a}_i, \vec{0} \rangle = 0 \tag{3}$$

Now we apply the distributive property of the inner product and the definition of orthonormal vectors,

 

$$\langle \vec{a}_i, \beta_1 \vec{a}_1 \rangle + \ldots + \langle \vec{a}_i, \beta_i \vec{a}_i \rangle + \ldots + \langle \vec{a}_i, \beta_N \vec{a}_N \rangle = 0 \quad (4)$$
$$0 + \ldots + \beta_i \langle \vec{a}_i, \vec{a}_i \rangle + \ldots + 0 = 0 \quad (5)$$
$$0 + \ldots + \beta_i \vec{a}_i^T \vec{a}_i + \ldots + 0 = 0 \quad (6)$$

Because $\vec{a}_i^T \vec{a}_i = 1$, $\beta_i = 0$ for the equation to hold. Then, since this is true for all $i$ from 1 to $N$, all the elements of the vector beta must be zero ($\vec{\beta} = \vec{0}$). Because $\vec{x} = \vec{0}$ implies $\vec{\beta} = \vec{0}$, the columns of $\mathbf{A}$ are linearly independent.
Now, we will show that any vector $\vec{x} \in \mathbb{R}^N$ can be represented as a linear combination of the columns of $\mathbf{A}$.

$$\vec{x} = \mathbf{A}\vec{\beta} = \beta_1 \vec{a}_1 + \ldots + \beta_N \vec{a}_N \quad (7)$$

Because we know that the $N$ columns of $\mathbf{A}$ are linearly independent, then there exists $\mathbf{A}^{-1}$. Applying the inverse to the equation above,

$$\mathbf{A}^{-1}\mathbf{A}\vec{\beta} = \mathbf{A}^{-1}\vec{x} \quad (8)$$
$$\vec{\beta} = \mathbf{A}^{-1}\vec{x}, \quad (9)$$

we find that there exists a unqiue $\beta$ that allow us to represent any $\vec{x}$ as a linear combination of the columns of $\mathbf{A}$.

(c) When $\mathbf{A} \in \mathbb{R}^{N \times M}$ and $N \geq M$ (i.e. tall matrices), show that if the matrix is orthonormal, then $\mathbf{A}^T\mathbf{A} = \mathbf{I}_{M \times M}$.
**Answer:** Want to show $\mathbf{A}^T\mathbf{A} = \mathbf{I}_{M \times M}$.

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \vec{a}_1^T \vec{a}_1 & \vec{a}_1^T \vec{a}_2 & \ldots & \vec{a}_1^T \vec{a}_n \\ \vec{a}_2^T \vec{a}_1 & \vec{a}_2^T \vec{a}_2 & \ldots & \vec{a}_2^T \vec{a}_n \\ \vdots & \vdots & & \vdots \end{bmatrix} = \mathbf{I}_{M \times M} \quad (10)$$

When $\vec{a}_i^T \vec{a}_i = \|\vec{a}_i\|^2 = 1$ and when $i \neq j$, $\vec{a}_i^T \vec{a}_j = 0$ because the column vectors are orthogonal.

(d) Again, suppose $\mathbf{A} \in \mathbb{R}^{N \times M}$ where $N \geq M$ is an orthonormal matrix. Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{A}$ is now $\mathbf{A}\mathbf{A}^T\vec{y}$.
**Answer:**
Starting with the result from part (a),

$$\mathbf{A}\hat{\vec{x}} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}, \quad (11)$$

we can apply the result from part (c),

$$\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y} = \mathbf{A}\mathbf{I}\mathbf{A}^T\vec{y} \quad (12)$$
$$= \mathbf{A}\mathbf{A}^T\vec{y} \quad (13)$$

(e) Given $\mathbf{A} \in \mathbb{R}^{N \times M} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and the columns of $\mathbf{A}$ are orthonormal, find the least squares solution to $\mathbf{A}\hat{\vec{x}} = \vec{y}$ where $\vec{y} = \begin{bmatrix} 5 & 12 & 7 & 8 \end{bmatrix}^T$.

**Answer:**

**Method 1:**

Since the columns of $\mathbf{A}$ are orthonormal, from part (d) we know that

$$\hat{\vec{x}} = \mathbf{A}^T \vec{y} = \begin{bmatrix} \langle \vec{a}_1, \vec{y} \rangle \\ \langle \vec{a}_2, \vec{y} \rangle \\ \langle \vec{a}_3, \vec{y} \rangle \end{bmatrix}.$$

Note that this is equivalent to projecting $\vec{y}$ onto each column of $\mathbf{A}$:

$\hat{x_1} = \frac{\langle \vec{a}_1, \vec{y} \rangle}{||\vec{a}_1||^2} = \langle \vec{a}_1, \vec{y} \rangle = 8$

$\hat{x_2} = \frac{\langle \vec{a}_2, \vec{y} \rangle}{||\vec{a}_2||^2} = \langle \vec{a}_2, \vec{y} \rangle = 7$

$\hat{x_3} = \frac{\langle \vec{a}_3, \vec{y} \rangle}{||\vec{a}_3||^2} = \langle \vec{a}_3, \vec{y} \rangle = \frac{17\sqrt{2}}{2}$

**Method 2 (Alternatively you can use the least squares formula):**

$$\hat{\vec{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y} = \left( \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \\ 7 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ \frac{17\sqrt{2}}{2} \end{bmatrix}$$

**2. Polynomial Fitting**

Let's try an example. Say we know that the output, $y$, is a quartic polynomial in $x$. This means that we know that $y$ and $x$ are related as follows:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

We're also given the following observations:

 

| $x$ | $y$ |
|---|---|
| 0.0 | 24.0 |
| 0.5 | 6.61 |
| 1.0 | 0.0 |
| 1.5 | −0.95 |
| 2.0 | 0.07 |
| 2.5 | 0.73 |
| 3.0 | −0.12 |
| 3.5 | −0.83 |
| 4.0 | −0.04 |
| 4.5 | 6.42 |

(a) What are the unknowns in this question?

**Solution:**

**Note to TAs:** Initiate a qualitative discussion with your students about what you're trying to solve for. Ask them if it makes sense that if we knew $a_0$, $a_1$, $a_2$, $a_3$, and $a_4$, we would exactly know the relationship between $x$ and $y$.

Doesn't that sound less scary than trying to "solve for the polynomial"? Great. Let's move on.

**Answer:**

The unknowns are $a_0$, $a_1$, $a_2$, $a_3$, and $a_4$.

(b) Can you write an equation corresponding to the first observation $(x_0, y_0)$, in terms of $a_0$, $a_1$, $a_2$, $a_3$, and $a_4$? What does this equation look like? Is it linear in the unknowns?

**Answer:**

Plugging $(x_0, y_0)$ into the expression for $y$ in terms of $x$, we get

$$24 = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + a_4 \cdot 0^4$$

You can see that this equation is linear in $a_0$, $a_1$, $a_2$, $a_3$, and $a_4$.

(c) Now, write a system of equations in terms of $a_0$, $a_1$, $a_2$, $a_3$, and $a_4$ using *all of the observations*.

**Solution:**

**Note to TAs:** Ask the students to write out the next equation using the second observation. You will get:

$$6.61 = a_0 + a_1 \cdot (0.5) + a_2 \cdot (0.5)^2 + a_3 \cdot (0.5)^3 + a_4 \cdot (0.5)^4$$

And the third:

$$0.0 = a_0 + a_1 \cdot (1) + a_2 \cdot 1^2 + a_3 \cdot 1^3 + a_4 \cdot 1^4$$

Do you see a pattern? Let's write the entire system of equations in terms of a matrix now.

$$\begin{bmatrix} 1 & 0 & 0^2 & 0^3 & 0^4 \\ 1 & 0.5 & (0.5)^2 & (0.5)^3 & (0.5)^4 \\ 1 & 1 & 1^2 & 1^3 & 1^4 \\ 1 & 1.5 & (1.5)^2 & (1.5)^3 & (1.5)^4 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 2.5 & (2.5)^2 & (2.5)^3 & (2.5)^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 3.5 & (3.5)^2 & (3.5)^3 & (3.5)^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 4.5 & (4.5)^2 & (4.5)^3 & (4.5)^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 24 \\ 6.61 \\ 0.0 \\ -0.95 \\ 0.07 \\ 0.73 \\ -0.12 \\ -0.83 \\ -0.04 \\ 6.42 \end{bmatrix}$$

If this is too much of a pain to write, don't write the entire thing out. At least write a few rows of it, though. It's useful to allude to the "data matrix" that we will see in the IPython notebook. Hopefully, when people see that, they should be able to connect to this example.

**Answer:**

Write the next equation using the second observation. You will now get:

$$6.61 = a_0 + a_1 \cdot (0.5) + a_2 \cdot (0.5)^2 + a_3 \cdot (0.5)^3 + a_4 \cdot (0.5)^4$$

And for the third:

$$0.0 = a_0 + a_1 \cdot (1) + a_2 \cdot 1^2 + a_3 \cdot 1^3 + a_4 \cdot 1^4$$

Do you see a pattern? Let's write the entire system of equations in terms of a matrix now.

$$\begin{bmatrix} 1 & 0 & 0^2 & 0^3 & 0^4 \\ 1 & 0.5 & (0.5)^2 & (0.5)^3 & (0.5)^4 \\ 1 & 1 & 1^2 & 1^3 & 1^4 \\ 1 & 1.5 & (1.5)^2 & (1.5)^3 & (1.5)^4 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 2.5 & (2.5)^2 & (2.5)^3 & (2.5)^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 3.5 & (3.5)^2 & (3.5)^3 & (3.5)^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 4.5 & (4.5)^2 & (4.5)^3 & (4.5)^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 24 \\ 6.61 \\ 0.0 \\ -0.95 \\ 0.07 \\ 0.73 \\ -0.12 \\ -0.83 \\ -0.04 \\ 6.42 \end{bmatrix}$$

(d) Finally, solve for $a_0$, $a_1$, $a_2$, $a_3$, and $a_4$ using IPython or any method you like. You have now found the quartic polynomial that best fits the data!

**Solution:**

**Note to TAs:** By now, students should be able to see how to calculate the least squares estimate. If you don't want to spend time on actually showing them how to calculate the solution, you don't have to. Reference the iPython file and run this for your students.

**Answer:**

Let $\mathbf{D}$ be the big matrix from the previous part.

$$\vec{a} = (\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T\vec{y} = \begin{bmatrix} 24.00958042 \\ -49.99515152 \\ 35.0039627 \\ -9.99561772 \\ 0.99841492 \end{bmatrix}$$

It turns out that the actual parameters for the polynomial equation were:

$$\vec{a} = \begin{bmatrix} 24 \\ -50 \\ 35 \\ -10 \\ 1 \end{bmatrix}$$

(Remember that our observations were noisy.)

Therefore, we have actually done pretty well with the least squares estimate!