EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 3B

1. Mechanical Inverses

For each sub-part below, determine whether or not the inverse of $\mathbf{A}$ exists.
If it exists, compute the inverse using Gauss-Jordan method.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

(b) (PRACTICE)
$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & -2 & 4 \end{bmatrix}$

(e) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

(f) (PRACTICE)

$$\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix}$$

2. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?
ii. What is the null space of $\mathbf{A}$? What is its dimension?
iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?
iv. Do the columns of $\mathbf{A}$ span $\mathbb{R}^2$? Do they form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

3. Helpful Guide – Reference Definitions

Vector spaces:

A *vector space* V is a set of elements that is 'closed' under vector addition and scalar multiplication and contains a zero vector. What does closed mean?

That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

More formally, a *vector space* (V, F) is a set of vectors V, a set of scalars F, and two operators that satisfy the following properties:

As a reminder, the mathematical notation $\forall\ \vec{v},\vec{u},\vec{w} \in V$ means *for all possible vectors $\vec{u},\vec{v},\vec{w}$ within the vector space V.*

- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w} \quad \forall \quad \vec{v},\vec{u},\vec{w} \in V$.
  - Commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u} \quad \forall \quad \vec{v},\vec{u} \in V$.
  - Additive Identity: There exists an additive identity $\vec{0} \in V$ such that $\vec{v}+\vec{0} = \vec{v} \quad \forall \quad \vec{v} \in V$.
  - Additive Inverse: For any $\vec{v} \in V$, there exists $-\vec{v} \in V$ such that $\vec{v}+(-\vec{v}) = \vec{0}$.
    We call $-\vec{v}$ the additive inverse of $\vec{v}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v} \quad \forall \quad \vec{v} \in V,\ \alpha,\beta \in F$.
  - Multiplicative Identity: There exists $1 \in F$ where $1 \cdot \vec{v} = \vec{v} \quad \forall \quad \vec{v} \in F$.
    We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v} \quad \forall \quad \alpha \in F$ and $\vec{u},\vec{v} \in V$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v} \quad \forall \quad \alpha,\beta \in F$ and $\vec{v} \in V$.

Subspaces:

A subset W of a *vector space* V is a *subspace* of V if the above conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace W.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

Basis:

A *basis* for a vector space or subspace is an *ordered set of linearly independent vectors* that *spans the vector space or subspace*.

Therefore, if we want to check whether a set of vectors $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ forms a basis for a vector space V, we check for two important properties:

(a) $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ is linearly independent.

(b) $\text{span}\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\} = V$

As we move along, we'll learn how to identify and construct a basis, and we'll also learn some interesting properties of bases.

Dimension:

The *dimension* of a vector space is the *minimum number* of vectors needed to span the entire vector space. That is, the dimension of a vector space equals the number of vectors in a basis for this vector space.