EECS 16A Spring 2021 — Designing Information Devices and Systems I

Discussion 5A

1. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

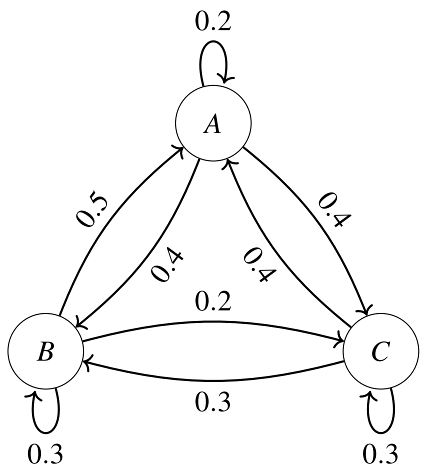

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix $\mathbf{T}$ representing the pumps system.

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of $\mathbf{T}$. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

(c) What does the magnitude of the other two eigenvalues λ_2 and λ_3 say about the steady state behavior of their associated eigenvectors?

(d) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

2. REVIEW – Change of Basis

When we first encountered vectors, we defined $\vec{r} \in \mathbb{R}^n$ as *an ordered list of 'n' numbers*. Now we can see them in a new light; as coordinates corresponding to some set of 'directions' (which are also vectors!). The default basis is the set of *elementary column vectors* $\vec{e}_j$:

$$\vec{r} \;=\; \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \;\equiv\; r_1\vec{e}_1 + r_2\vec{e}_2 + \ldots + r_n\vec{e}_n \;=\; r_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + r_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \vec{r} \;=\; \mathbf{I}\,\vec{r}$$

Now suppose we'd like to use a new set of basis vectors $\mathbb{V} := \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, which we can expand in a similar fashion!

$$\vec{r} \equiv\; r_1^{(v)}\vec{v}_1 + r_2^{(v)}\vec{v}_2 + \ldots + r_n^{(v)}\vec{v}_n \;=\; \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} r_1^{(v)} \\ r_2^{(v)} \\ \vdots \\ r_n^{(v)} \end{bmatrix} \;=\; \mathbf{V}\,\vec{r}^{(v)}$$

NOTE! Since we have required that $\mathbb{V}$ is a basis, we know (1) all vectors $\vec{v}_j$ are linearly independent from each other, and (2) it is a minimal set to span $\mathbb{R}^n$. From these properties we conclude that $\mathbf{V}$ is square and invertible! Thus we can identify the vector $\vec{r}^{(v)}$ expressed in the $\mathbb{V}$ basis from the original form of $\vec{r}$ using

$$\vec{r}^{(v)} \;=\; \mathbf{V}^{-1}\,\vec{r}\;.$$

But what if (instead of the elementary basis) we had started in some other basis $\mathbb{U} := \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$, could we apply a transformation to $\vec{r}$ to map from the $\mathbb{U}$ basis to the $\mathbb{V}$ basis?
Yes we can!

In fact, it follows quite naturally from our previous reasoning, just applied to a second basis.

$$\begin{aligned} \vec{r} \;=\;\; & r_1\,\vec{e}_1 + r_2\,\vec{e}_2 + \ldots + r_n\,\vec{e}_n \;=\; && \mathbf{I}\,\vec{r} \\ & r_1^{(v)}\vec{v}_1 + r_2^{(v)}\vec{v}_2 + \ldots + r_n^{(v)}\vec{v}_n \;=\; && \mathbf{V}\,\vec{r}^{(v)} \\ & r_1^{(u)}\vec{u}_1 + r_2^{(u)}\vec{u}_2 + \ldots + r_n^{(u)}\vec{u}_n \;=\; && \mathbf{U}\,\vec{r}^{(u)} \end{aligned}$$

From these equalities we can derive the transformation from $\vec{r}^{(v)}$ into $\vec{r}^{(u)}$ using $\mathbf{W} = \mathbf{V^{-1}}\,\mathbf{U}$ as shown below:

$$\vec{r}^{(v)} \;=\; \mathbf{V}^{-1}\mathbf{U}\;\vec{r}^{(u)}$$

3. Coordinate Change Examples

For the following sub-parts, two sets of basis vectors are specified $\mathbb{U} := \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and $\mathbb{V} := \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Provided the vector $\vec{r}$ expressed in the $\mathbb{U}$ basis $\vec{r}^{(u)}$, identify the matrix describing the transformation $\vec{r}^{(u)} \rightarrow \vec{r}^{(v)}$:

(a) Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{r}^{(v)} = \mathbf{T}\vec{r}^{(u)}$ where $\vec{r}^{(u)}$ contains the coordinates of a vector in a basis of the columns of $\mathbf{U}$ and $\vec{r}^{(v)}$ is the coordinates of the same vector in the basis of the columns of $\mathbf{V}$.
Let $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{r}^{(v)}$. Repeat this for $\vec{r}^{(u)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now let $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What is $\vec{r}^{(v)}$?

(b) Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{r}^{(v)} = \mathbf{T}\vec{r}^{(u)}$.
Let $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and compute $\vec{r}^{(v)}$. Repeat this for $\vec{r}^{(u)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) Let us return to thinking in general about two sets of basis vectors for $\mathbb{R}^n$, $\mathbb{U} := \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and $\mathbb{V} := \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Conversely, what is the coordinate transformation from $\vec{r}^{(v)}$ to $\vec{r}^{(u)}$? Find $\mathbf{Q}$ in terms of the two basis sets such that $\vec{r}^{(u)} = \mathbf{Q}\vec{r}^{(v)}$:

(d) **(PRACTICE)** Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{r}^{(v)} = \mathbf{T}\vec{r}^{(u)}$. Let $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{r}^{(v)}$. Repeat this for $\vec{r}^{(u)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{r}^{(u)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{r}^{(v)}$?