EECS 16A Designing Information Devices and Systems I
Spring 2022 Discussion 6A

1. True or False?

For each of the following subparts below, prove whether the statement is True or False.

(a) There exists an invertible $n \times n$ matrix A for which $A^2 = 0$.

(b) If A is an invertible $n \times n$ matrix, then for all vectors $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ has a unique solution.

(c) If A and B are invertible $n \times n$ matrices, then the product AB is invertible.

(d) The two vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ form a basis for the subspace, span$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$.

(e) The dimension of the subspace, span$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$, is 3.

(f) A set of n linearly dependent vectors in $\mathbb{R}^n$ can span $\mathbb{R}^n$.

2. Are eigenvectors linearly independent?

Suppose we have a square matrix $\mathbf{A}^{n \times n}$ with 'n' distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ (meaning that $\lambda_i \neq \lambda_j$ when $i \neq j$) and 'n' corresponding eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Prove that any two eigenvectors $\vec{v}_i, \vec{v}_j$ (for $i \neq j$) are linearly independent.

HINT: Begin proof by contradiction: Suppose that $\vec{v}_i$ and $\vec{v}_j$ correspond to distinct eigenvalues, so that $(\lambda_i - \lambda_j) \neq 0$, and are linearly dependent. Show this leads to a nonsensical equality after applying $\mathbf{A}$. *(Please ask the staff for guidance if this hint is too vague!)*

3. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

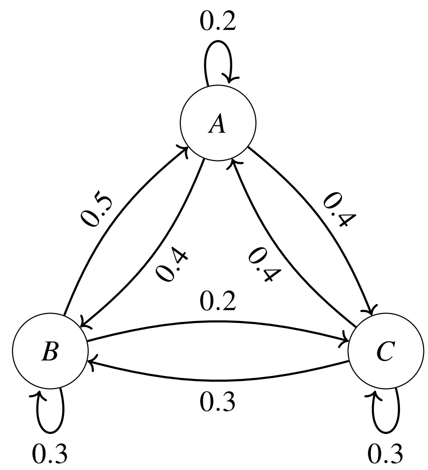

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix $\mathbf{T}$ representing the pumps system.

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of $\mathbf{T}$. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

(c) What does the magnitude of the other two eigenvalues λ_2 and λ_3 say about the steady state behavior of their associated eigenvectors?

(d) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?