Dis 1B EECS16A

tinyurl.com/miyuki-feedback

1. Finding The Bright Cave

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoul fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers, but they don't know any linear algebra – and they need your help.

Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

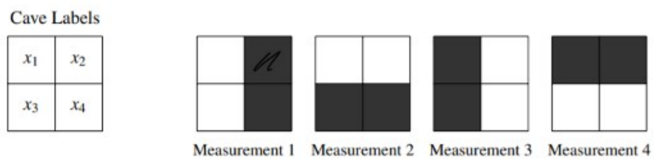

Figure 1: Four image masks.

(a) Let $\vec{x}$ be the four-element vector that represents the magnitude of light emanating from the four cave entrances. Write a matrix **K** that performs the masking process in Figure 1 on the vector $\vec{x}$, such that $\mathbf{K}\vec{x}$ is the result of the four measurements.

$\vec{m} = K\vec{x}$

unknowns

① $m_1 = x_1 + 0 + x_3 + 0$
② $m_2 = x_1 + x_2 + 0 + 0$
③ $m_3 = 0 + x_2 + 0 + x_4$
④ $m_4 = 0 + 0 + x_3 + x_4$

$\Longleftrightarrow$

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(b) Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not?

i.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_1 - m_2 + m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right]$$

$$\xrightarrow{R_4 \leftarrow R_4 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_1 - m_2 + m_3 \\ 0 & 0 & 0 & 0 & m_4 - m_1 + m_2 - m_3 \end{array}\right]$$

ref

unique? no

sol case? inf if $m_4 - m_1 + m_2 - m_3 = 0$

no sol " " $\neq 0$

* just to check if it's unique could've done

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ii. Ex: $x + y = 1$, $2x + 2y = 2$ } not unique

① + ③ = ② + ④

④ = ① + ③ − ②

eqn ④ is not providing any new information

* one of these eqns is not providing new info

Ex: $x + y = 1$
$2x + 2y = 2$
$x - y = 3$

(c) Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However, her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

x_1	x_2
50%	100%
100%	50%
x_3	x_4

m_5

Does this additional measurement give them enough information to solve the problem? Why or why not?

$$m_5 = 0.5x_1 + x_2 + x_3 + 0.5x_4$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 0.5 & 1 & 1 & 0.5 & m_5 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -m_5 + \frac{m_3}{2} + \frac{m_2}{2} + m_1 \\ 0 & 1 & 0 & 0 & m_5 - \frac{m_3}{2} + \frac{m_2}{2} - m_1 \\ 0 & 0 & 1 & 0 & m_5 - \frac{m_3}{2} - \frac{m_2}{2} \\ 0 & 0 & 0 & 1 & -m_5 + \frac{3m_3}{2} - \frac{m_2}{2} + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right]$$

unique? maybe

if $m_4 - m_3 + m_2 - m_1 = 0$ unique

$\neq 0$ no sol

2. Proofs

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is **linearly dependent** if there exists constants $c_1, c_2, \dots\ c_n$ such that $\sum_{i=1}^{i=n} c_i\vec{v}_i = \vec{0}$ and at least one c_i is non-zero.

This condition intuitively states that it is possible to express any vector from the set in terms of the others.

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$

→ this set of vectors is lin dep if we can find some $c_1, \dots, c_n$ where at least one $c_i \neq 0$

Ex: $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix} \right\}$

$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}2\\3\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

could choose $c_1 = c_2 = c_3 = 0$ to make this true but can we find any other combo of c_1, c_2, c_3 to make this true?

$c_1 = -2,\ c_2 = -3,\ c_3 = 1$

since yes → the set of vectors is lin dep

Strategies for Proofs

- write out mathematical defn for what you know & what you want to show
- try simple examples to find patterns
- manipulate the defns to get from what you know to what you want to show

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$

know: $\vec{x} \neq \vec{0} \rightarrow \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} \neq \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix}$ $\quad A\vec{x} = \vec{0}$

show: cols of A are lin dep

$c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n = \vec{0}$ at least one $c_i \neq 0$

$A\vec{x} = \vec{0} \quad \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix}$

looks similar

$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{0}$

since $\vec{x} \neq \vec{0}$ there exists some $x_i \neq 0$

↳ cols of A are lin dep

(b) For $\mathbf{A} \in \mathbb{R}^{m \times n}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Know: $\vec{x}_1 \neq \vec{x}_2 \quad A\vec{x}_1 = \vec{b} \quad A\vec{x}_2 = \vec{b}$

Show: cols of A are lin dep

$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = \vec{0}$ where some $c_i \neq 0$

Ex: $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = 2 \quad x_2 = -1$

$x_1 = -2 \quad x_2 = 1$

$\vec{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b}$

$A(\vec{x}_1 - \vec{x}_2) = \vec{0}$

$A\vec{y} = \vec{0}$

$* \ \vec{x}_1 - \vec{x}_2$

$\vec{y} = \vec{x}_1 - \vec{x}_2 \neq \vec{0}$

because $\vec{x}_1 \neq \vec{x}_2$

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{0}$$

$y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n = \vec{0}$ because $\vec{y} \neq 0$ there is some $y_i \neq 0$ $\rightarrow$ cols of A are lin dep

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix for which there exists a non-zero $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{y} = \vec{0}$. Let $\vec{b} \in \mathbb{R}^m$ be some non zero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

Know: $\vec{y} \neq \vec{0} \quad A\vec{y} = \vec{0}$

Show: $A\vec{x} = \vec{b}$ has inf sol

$A\vec{x}_1 = \vec{b} \quad A\vec{x}_2 = \vec{b} \quad \vec{x}_1 \neq \vec{x}_2$

$A\vec{x}_1 = \vec{b} \qquad A\vec{y} = \vec{0}$

$A\vec{x}_1 + A\vec{y} = \vec{b} + \vec{0}$

$A(\underbrace{\vec{x}_1 + \vec{y}}_{\vec{x}_2}) = \vec{b}$

$\vec{x}_1 \neq \vec{x}_2$ because $\vec{y} \neq \vec{0}$

$\vec{x}_1 \neq \vec{x}_2 \neq \vec{x}_3$

$2A\vec{y} = A(2\vec{y})$

$A\vec{x}_1 + A(2\vec{y}) = \vec{b} + 0$

$A(\underbrace{\vec{x}_1 + 2\vec{y}}_{\vec{x}_3}) = \vec{b}$

inf sol of form

$\vec{x} = \vec{x}_1 + c\vec{y}$

$\nearrow$ const