EECS 16A Designing Information Devices and Systems I
Spring 2021 Homework 4

This homework is due February 19, 2021, at 23:59.
Self-grades are due February 22, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit each file to its respective assignment on Gradescope.

Study group task: We recommend you use some time during your study group to develop a plan for studying for the first midterm, which is coming soon. Discuss how you can support each other in your studying. Do you want to have an extra meeting to discuss some harder concepts? If you are not working with a group, you may make your plan alone. Make a list of what you want to review — this should include lectures, discussion problems, homework problems, and notes.

1. Reading Assignment

For this homework, please review Note 5 and read Notes 6, 7. The notes 5 and 6 provide an overview of multiplication of matrices with vectors, by considering the example of water reservoirs and water pumps, and matrix inversion. Note 7 provides an introduction to vector spaces. You are always welcome and encouraged to read beyond this as well. Note 8 discusses column spaces and nullspaces, so it might be useful to read that for this homework as well.

You have seen in Note 5 that the pump system can be represented by a state transition matrix. What constraint must this matrix satisfy in order for the pump system to obey water conservation?

2. Feedback on your study groups

Please help us understand how your study groups are going! Fill out the following survey to help us create better matchings in the future. In case you have not been able to connect with a study group, or would like to try a new study group, there will be an opportunity for you to request a new study group as well in this form.

https://forms.gle/GAovoM3WxYYZYc5PA

To get full credit for this question you must (1) fill out the survey (it will record your email) and (2) indicate in your homework submission that you filled out the survey.

3. Mechanical Inverses

***Learning Objectives**: Matrices represent linear transformations, and their inverses represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.*

For each of the following values of matrix A:

i Find the inverse, $\mathbf{A}^{-1}$, if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.

ii **For parts (a)-(d)**, in addition to finding the inverse (if it exists), describe how the matrix $\mathbf{A}$ transforms an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

For example, if $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$, then $\mathbf{A}$ could scale $\begin{bmatrix} x \\ y \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. If $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$, then $\mathbf{A}$ could reflect $\begin{bmatrix} x \\ y \end{bmatrix}$ across the x axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

iii **For parts (a)-(d)**, if we use $\mathbf{A}$ to geometrically transform $\begin{bmatrix} x \\ y \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$, **is it possible to reverse the transformation geometrically,** i.e. is it possible to retrieve $\begin{bmatrix} x \\ y \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Assume $\cos\theta \neq 0$. *Hint:* $\cos^2\theta + \sin^2\theta = 1$.

(e) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

(f) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

(g) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$

(h) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(i) **(OPTIONAL)**

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix? Hint 2: We're reasonable people!

4. Properties of Pump Systems

Learning Objectives: *This problem illustrates how matrices and vectors can be used to represent linear transformations.*

Throughout this problem, we will consider a system of reservoirs connected to each other through pumps. An example system is shown below in Figure 1, represented as a graph. Each node in the graph is marked with a letter and **represents a reservoir**. Each arrow in the graph represents a pump which moves a fraction of the water from one reservoir to the next at every time step. The **fraction of water** is written on top of the arrow.

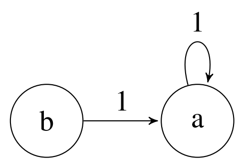


Figure 1: Pump system

(a) For the system of pumps shown in Figure 1, find the associated state transition matrix. In other words, find the matrix $\mathbf{A}$ such that:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n], \text{ where } \vec{x}[n] = \begin{bmatrix} x_a[n] \\ x_b[n] \end{bmatrix}$$

$x_a[n]$ and $x_b[n]$ represent the amount of water in reservoir a and b, respectively, at time step n.

(b) Let us assume that at time step 0, the reservoirs are initialized to the following water levels: $x_a[0] = 0.5, x_b[0] = 0.5$. In a completely alternate universe, the reservoirs are initialized to the following water levels: $x_a[0] = 0.3, x_b[0] = 0.7$. For both initial states, what are the water levels at timestep 1 ($\vec{x}[1]$)? Use your answer from part (a) to compute your solution.

(c) If you observe the reservoirs at timestep 1, i.e if you know $\vec{x}[1]$, can you figure out what the initial ($\vec{x}[0]$) water levels were? Why or why not?

(d) Now let us generalize what we observed. Say there is a transition matrix $\mathbf{A}$ representing a pump system, and there exist two distinct initial state vectors/water levels: $\vec{x_u}[0]$ and $\vec{x_v}[0]$, that lead to the same state vector $\vec{x}[1]$ after $\mathbf{A}$ acts on them. You do not know which of the two initial state vectors you started in. Can you decide which initial state you started in by observing $\vec{x}[1]$? What does this say about the matrix $\mathbf{A}$?

(e) Now, we want to prove the following theorem in a step-by-step fashion.

Theorem: Consider a system consisting of k reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then the total amount of water at timestep $(n+1)$ will also be s.

i. Since the problem does not specify the transition matrix, let us start with a transition matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and a state vector $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$ (this is an example when $k = 2$). In general, it is helpful to write as much out mathematically as you can in proofs. It can also be helpful to draw the transition graph. Write out what is "known", i.e. ALL the information that is given to you in the theorem statement *in mathematical form*.

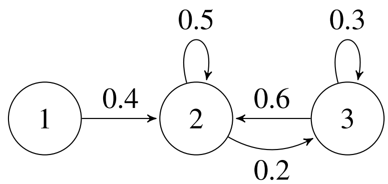
ii. Now write out what is to be proved *in mathematical form*.

iii. Prove the statement for the case of two reservoirs.

iv. Now use what you learned to generalize to the case of k reservoirs. *Hint:* Think about $\mathbf{A}$ in terms of its columns, since you have information about sum of each column.

(f) Set up the state transition matrix $\mathbf{A}$ for the system of pumps shown below. Compute the sum of the entries of each column of the state transition matrix. Are the sums greater than/less than/equal to 1? Explain what this $\mathbf{A}$ matrix physically implies about how the total amount of water in this system changes over time.

Note: If there is no "self-arrow/self-loop," you can interpret it as a self-loop with weight 0, i.e. no water returns.



5. [OPTIONAL] Mechanical Basis

Learning Objectives: *Determining the basis of a vector space.*

(a) Let vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3 \in \mathbb{R}^4$:

$$\vec{v}_1 = \begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}2\\0\\3\\0\end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form a basis for the vector space $\mathbb{R}^4$? Justify your answer.

(b) Let $\vec{x} = \begin{bmatrix}5\\3\\6\end{bmatrix}$. Given a new set of vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}3\\3\\0\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}2\\0\\2\end{bmatrix}, \vec{v}_5 = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ be a basis for $\mathbb{R}^3$? If so, express $\vec{x}$ as a linear combination of these basis vectors.

If the set of five vectors cannot form a basis for $\mathbb{R}^3$, choose a new basis including $\vec{v}_1$, $\vec{v}_2$ and any number of additional vectors from the set. Then express $\vec{x}$ as a linear combination of the newly constructed basis vectors.

(c) Let $\vec{x} = \begin{bmatrix}3\\5\\4\end{bmatrix}$. Given a new set of vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix}1\\3\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}2\\7\\1\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}2\\0\\2\end{bmatrix}, \vec{v}_5 = \begin{bmatrix}1\\6\\3\end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ be a basis for $\mathbb{R}^3$? If so, express $\vec{x}$ as a linear combination of these basis vectors.

If the set of five vectors cannot form a basis for $\mathbb{R}^3$, choose a new basis including $\vec{v}_1$, $\vec{v}_2$ and any number of additional vectors from the set. Then express $\vec{x}$ as a linear combination of the newly constructed basis vectors.

6. Finding Null Spaces and Column Spaces

Learning Objectives*: Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

Definition (Null space): The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as $\mathrm{N}(\mathbf{A})$ and the definition can be written in set notation as:

$$\mathrm{N}(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the set of $\mathbf{A}$'s columns. The column space can be notated as $\mathrm{C}(\mathbf{A})$ or $\mathrm{range}(\mathbf{A})$ and the definition can be written in set notation as:

$$\mathrm{C}(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

Definition (Dimension): The dimension of a vector space is the number of basis vectors - i.e. the minimum number of vectors required to span the vector space.

(a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3 \times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of $\mathrm{C}(\mathbf{A})$?

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $\mathrm{C}(\mathbf{A})$ (i.e. a basis for $\mathrm{C}(\mathbf{A})$). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of $\mathrm{C}(\mathbf{A})$?

Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.

(c) Find a *minimum* set of vectors that span $\mathrm{N}(\mathbf{A})$ (i.e. a basis for $\mathrm{N}(\mathbf{A})$), where $\mathbf{A}$ is the same matrix as in part (b). What is the dimension of $\mathrm{N}(\mathbf{A})$?

(d) Find the sum of the dimensions of $\mathrm{N}(\mathbf{A})$ and $\mathrm{C}(\mathbf{A})$. What do you notice about this sum in relation to the dimensions of $\mathbf{A}$?

(e) Now consider the new matrix, $\mathbf{B} = \mathbf{A}^T$,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $C(\mathbf{B})$ (i.e. a basis for $C(\mathbf{B})$). What is the minimum number of vectors required to span the $C(\mathbf{B})$?

(f) You are given the following matrix $\mathbf{G}$. Find a *minimum* set of vectors that span $N(\mathbf{G})$, i.e. a basis for $N(\mathbf{G})$.

$$\mathbf{G} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

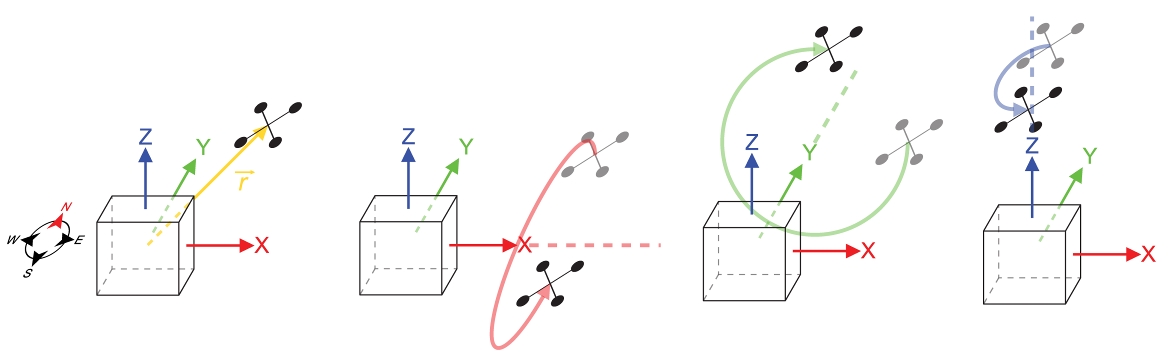
(g) (**OPTIONAL**) For the following matrix $\mathbf{D}$, find $C(\mathbf{D})$ and its dimension, and $N(\mathbf{D})$ and its dimension.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

7. [OPTIONAL] Quadcopter Transformations

Learning Objectives: *Linear algebra is often used to represent transformations in robotics. This problem introduces some of the basic uses of transformations.*

Vijay and his colleagues are interested in developing a communication link using a laser to control the location of a quadrotor. Consider a vector $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ representing the location of the quadcopter relative to the origin. The quadcopter is only capable of three different maneuvers relative to the origin. The maneuvers are rotations about the x, y, and z axes. For perspective, the positive x-axis points east, the positive y-axis points north, and the positive z-axis points up towards the sky. The figures below illustrate the quadcopter and these maneuvers.



We can represent each of these rotations, that are linear transformations, as matrices that operate on the location vector of the quadcopter, $\vec{r}$, to position it at its new location. The matrices $\mathbf{R_x}(\theta)$, $\mathbf{R_y}(\psi)$, and $\mathbf{R_z}(\phi)$ represent rotations about the x-axis, y-axis, and z-axis, respectively. The matrices are:

$$\mathbf{R_x}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \mathbf{R_y}(\psi) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}, \mathbf{R_z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

.

(a) Vijay wants to make the quadcopter rotate first by 30° about the x-axis, and then by 60° about the z-axis. Use $\mathbf{R_x}(\theta)$, $\mathbf{R_y}(\psi)$, and $\mathbf{R_z}(\phi)$ to construct **a single matrix that performs the operations in the specified order**. Show the matrix operations and calculations by hand.

(b) Vijay accidentally punched in the two rotation commands in reverse. The 60° rotation about the z-axis occurred before the 30° rotation about the x-axis. Use $\mathbf{R_x}(\theta)$, $\mathbf{R_y}(\psi)$, and $\mathbf{R_z}(\phi)$ to construct **a single matrix that performs the operations in the accidentally reversed order**. Show the matrix operations and calculations by hand.

(c) Say the quadcopter was initially positioned at $\vec{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

i. Where did Vijay intend for the quadcopter to end up? Use your result from part (a) to find this out.

ii. Where did the quadcopter actually end up with the accidentally reversed order of the rotations? Use your result from part (b) to find this out.

iii. Did the quadcopter end up where it was supposed to go?

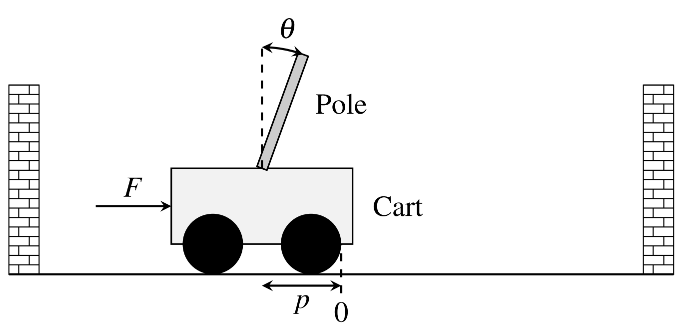
8. Segway Tours

Learning Objective: *The learning objective of this problem is to see how the concept of span can be applied to control problems. If a desired state vector of a linear control problem is in a span of a particular set of vectors, then the system may be steered to reach that particular vector using the available inputs.*

Your friends have decided to start a new SF tour business, and you suggest they use segways. They become intrigued by your idea and asks you how a segway works. A segway is essentially a stand on two wheels.

The segway works by applying a force (through the spinning wheels) to the base of the segway. This controls both the position on the segway and the angle of the stand. As the driver pushes on the stand, the segway tries to bring itself back to the upright position, and it can only do this by moving the base.

Is it possible for the segway to be brought upright and to a stop from any initial configuration? There is only one input (force) used to control two outputs (position and angle). You talk to a friend who is GSIing EE128, and she tells you that a segway can be modeled as a cart-pole system.



A cart-pole system can be fully described by its position p, velocity $\dot{p}$, angle θ, and angular velocity $\dot{\theta}$. We write this as a "state vector", $\vec{x}$:

$$\vec{x} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix}.$$

The input to this system is a scalar quantity $u[n]$ at time n, which is the force F applied to the cart (or base of the segway).[1]

The cart-pole system can be represented by a linear model:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n] + \vec{b}u[n], \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{4\times 4}$ and $\vec{b} \in \mathbb{R}^{4\times 1}$.

The control $u[n]$ allows us to move the state ($\vec{x}$) in the direction of $\vec{b}$. So, if $u[n] = 2$, we move the state by $2\vec{b}$ at time n, and so on. We can choose different controls at different times.

The model tells us how the state vector, $\vec{x}$, will evolve over time as a function of the current state vector and control inputs.

You look at this general linear system and try to answer the following question: Starting from some initial state $\vec{x}_0$, can we reach a final desired state, $\vec{x}_f$, in N steps?

The challenge seems to be that the state is four-dimensional and keeps evolving and that we can only apply a one-dimensional (scalar) control at each time. Typically, to set the values of four variables to desired quantities, you would need four inputs. Can you do this with just one input?

We will solve this problem by walking through several steps.

(a) Express $\vec{x}[1]$ in terms of $\vec{x}[0]$ and the input $u[0]$.

(b) i. Express $\vec{x}[2]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$ and $u[1]$.

ii. Then express $\vec{x}[3]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, and $u[2]$.

iii. Finally express $\vec{x}[4]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, $u[2]$, and $u[3]$.
Your expressions can have other relevant variables (e.g. A, $\vec{b}$ etc) and mathematical operators.

(c) Now, generalize the pattern you saw in the earlier part to write an expression for $\vec{x}[N]$ in terms of $\vec{x}[0]$ and the inputs from $u[0], \ldots, u[N-1]$. **Your expression can have other relevant variables (e.g. A, $\vec{b}$ etc) and mathematical operators.**

For the next four parts of the problem, you are given the matrix $\mathbf{A}$ and the vector $\vec{b}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.05 & -0.01 & 0 \\ 0 & 0.22 & -0.17 & -0.01 \\ 0 & 0.10 & 1.14 & 0.10 \\ 0 & 1.66 & 2.85 & 1.14 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0.01 \\ 0.21 \\ -0.03 \\ -0.44 \end{bmatrix}$$

[1] You might note that velocity and angular velocity are derivatives of position and angle respectively. Differential equations are used to describe continuous time systems, which you will learn more about in EECS 16B. But even without these techniques, we can still approximate the solution to be a continuous time system by modeling it as a discrete time system where we take very small steps in time. We think about applying a force constantly for a given finite duration and we see how the system responds after that finite duration.

Assume the cart-pole starts in an initial state $\vec{x}[0] = \begin{bmatrix} -0.3853493 \\ 6.1032227 \\ 0.8120005 \\ -14 \end{bmatrix}$, and you want to reach the desired state $\vec{x}_f = \vec{0}$ using the control inputs $u[0], u[1], \ldots$ etc. The state vector $\vec{x}_f = \vec{0}$ corresponds to the cart-pole (or segway) being upright and stopped at the origin. **Reaching $\vec{x}_f = \vec{0}$ in N steps means that, given that we start at $\vec{x}[0]$, we can find control inputs ($u[0], u[1], \ldots$ etc), such that we get $\vec{x}[N]$ (i.e. state vector at Nth time step) equal to $\vec{x}_f = \vec{0}$.**

Note: Please use the Jupyter notebook to solve parts (d) - (g) of the problem. You may use the function we provided `gauss_elim(matrix)` to help you find the **upper triangular form** of matrices. An example of Gaussian Elimination using (`gauss_elim(matrix)`) is provided in the Jupyter notebook under section **Example Usage of gauss_elim**. You may also use the function (`np.linalg.solve`) to solve the equations.

(d) Can you reach $\vec{x}_f$ in *two* time steps? Show work to justify your answer. You should manipulate the equations on paper, but then use the Jupyter notebook for numerical computations.
(*Hint: Express $\vec{x}[2] - \mathbf{A}^2\vec{x}[0]$ in terms of the inputs $u[0]$ and $u[1]$. Then determine if the system of equations can be solved to obtain $u[0]$ and $u[1]$. If we obtain valid solutions for $u[0]$ and $u[1]$, then we can say we will reach $\vec{x}_f$ in two time steps. Use the notebook to see if the system of equations can be solved.*)

(e) Can you reach $\vec{x}_f$ in *three* time steps? Show work to justify your answer. You should manipulate the equations on paper, but then use the Jupyter notebook for numerical computations.
(*Hint: Similar to the last part, express $\vec{x}[3] - \mathbf{A}^3\vec{x}[0]$ in terms of the inputs $u[0]$, $u[1]$ and $u[2]$. Then determine if we can obtain valid solutions for $u[0]$, $u[1]$ and $u[2]$.*)

(f) Can you reach $\vec{x}_f$ in *four* time steps? Show work to justify your answer. You should manipulate the equations on paper, but then use the Jupyter notebook for numerical computations. (*Use the hints from the last two parts.*)

(g) If you have found that you can get to the final state in 4 time steps, find the required correct control inputs, i.e. $u[0]$, $u[1]$, $u[2]$ and $u[3]$, using Jupyter and verify the answer by entering these control inputs into the **Plug in your controller** section of the code in the Jupyter notebook. You need to just show that you reached the desired final state $\vec{x}_f$ by plugging in the control inputs. The code has been already written to simulate this system.
Suggestion: See what happens if you enter all four control inputs equal to 0. This gives you an idea of how the system naturally evolves!

(h) Let us reflect on what we just did. Recall the system we have:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n] + \vec{b}u[n].$$

The control allows us to move the state at time step $n+1$ by $u[n]$ in direction $\vec{b}$, remember $u[n] \in \mathbb{R}$ is just a scalar. We know from part (c) that:

$$\vec{x}[2] = \mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1].$$

Again, here $u[0], u[1] \in \mathbb{R}$ can be thought of as arbitrary scalars, and $\mathbf{A}\vec{b}u[0] + \vec{b}u[1]$ can be thought of as the set of all linear combinations of the vectors $\vec{b}$ and $\mathbf{A}\vec{b}$. Using this observation, can you express the possible states you can arrive at in two time steps using the span of exactly *two* vectors plus a vector offset?

(i) Let's try to generalize the idea in the previous part. Express the states you can reach in N timesteps as a span of some vectors plus a vector offset. (Hint: Consider the direction that each control input $u[0],\ldots,u[N-1]$ can move $\vec{x}[N]$ by.)

(j) **(OPTIONAL)** Now say you wanted to reach anywhere in $\mathbb{R}^4$, i.e. $\vec{x}_f$ is an unspecified vector in $\mathbb{R}^4$. Under what conditions can you guarantee that you can "reach" $\vec{x}_f$ from any $\vec{x}_0$ in N time steps? Wouldn't this be cool?

9. Page Rank

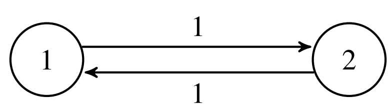
Learning Goal: *This problem highlights the use of transition matrices in modeling dynamical linear systems. Predictions about the steady state of a system can be made using the eigenvalues and eigenvectors of this matrix.*

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the setting of web traffic where the dynamical system can be described with a directed graph, also known as state transition diagram.

As we have seen in lecture and discussion the "transition matrix", $\mathbf{T}$, can be constructed using the state transition diagram, as follows: entries t_{ji}, represent the *proportion* of the people who are at website i that click the link for website j.

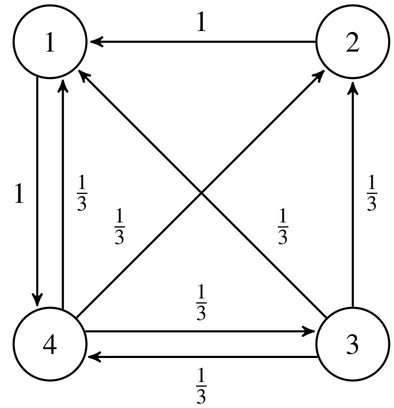
The **steady-state frequency** (i.e. fraction of visitors in steady-state) for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph. Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When the elements of this eigenvector are made to **sum to one** (to conserve population), the i^{th} element of the eigenvector will correspond to the fraction of people on the i^{th} website.

(a) For graph A shown below, what are the steady-state frequencies i.e. fraction of visitors in steady-state for the two webpages? Graph A has weights in place to help you construct the transition matrix. Remember to ensure that your steady state-frequencies sum to 1 to maintain conservation.



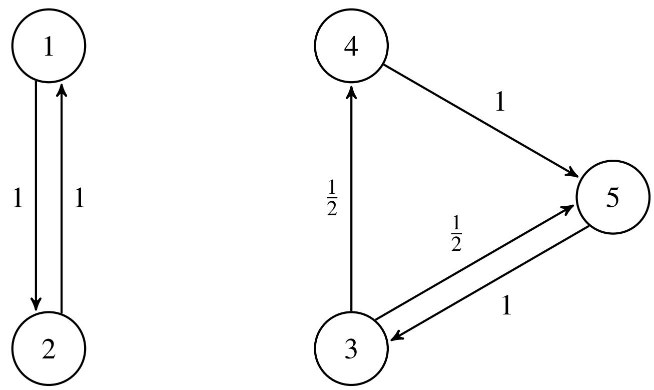
Graph A

(b) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this. It may be helpful to consult the Python documentation for `numpy.linalg.eig` to understand what this function does and what it returns. Graph B is shown below, with weights in place to help you construct the transition matrix.



Graph B

(c) Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C? Again, graph C with weights in place is shown below. You may use IPython to compute the eigenvalues and eigenvectors again.



Graph C

10. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.