EECS 16A Designing Information Devices and Systems I
Spring 2022 Homework 6

This homework is due Friday, March 4, 2021, at 23:59.
Self-grades are due Monday, March 7, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit the file to the appropriate assignment on Gradescope.

1. **Reading Assignment**

   For this homework, please review and read Note 11A/B, which introduces the basics of circuit analysis and node voltage analysis. You are always welcome and encouraged to read beyond this as well.

2. **Page Rank**

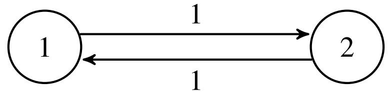
   Learning Goal: *This problem highlights the use of transition matrices in modeling dynamical linear systems. Predictions about the steady state of a system can be made using the eigenvalues and eigenvectors of this matrix.*

   In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the setting of web traffic where the dynamical system can be described with a directed graph, also known as state transition diagram.

   As we have seen in lecture and discussion the "transition matrix", $\mathbf{T}$, can be constructed using the state transition diagram, as follows: entries t_{ji} represent the *proportion* of the people who are at website i that click the link for website j.

   The **steady-state frequency** (i.e. fraction of visitors in steady-state) for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph. Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When the elements of this eigenvector are made to **sum to one** (to conserve population), the i^{th} element of the eigenvector will correspond to the fraction of people on the i^{th} website.

   (a) For graph A shown below, what are the steady-state frequencies (i.e. fraction of visitors in steady-state) for the two webpages? Graph A has weights in place to help you construct the transition matrix. Remember to ensure that your steady state-frequencies sum to 1 to maintain conservation.



Graph A

Solution:

To determine the steady-state frequencies for the two pages, we need to find the appropriate eigenvector of the transition matrix. In this case, we are trying to determine the proportion of people who would be on a given page at steady state.

The transition matrix of graph A:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

To determine the eigenvalues of this matrix:

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = 0 \tag{2}$$

$\lambda = 1, -1$. The steady state vector is the eigenvector that corresponds to $\lambda = 1$. To find the eigenvector,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3}$$

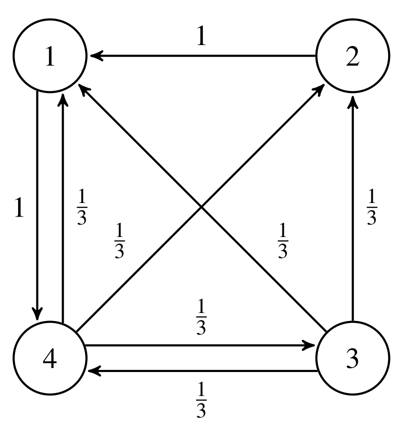
The sum of the values of the vector should equal 1 since the number of people is conserved, so our conditions are:

$$v_1 + v_2 = 1$$
$$v_1 = v_2$$

The steady-state frequency eigenvector is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and each webpage has a steady-state frequency of 0.5.

(b) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this. Graph B is shown below, with weights in place to help you construct the transition matrix.
Hint: `numpy.linalg.eig` returns eigenvectors and eigenvalues. The eigenvectors are arranged in a matrix in *column-major* order. In other words, given eigenvectors
$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$
NumPy will return:

$$\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \tag{4}$$



Graph B

Solution:

To determine the steady-state frequencies, we need to create the transition matrix $\mathbf{T}$ first.

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

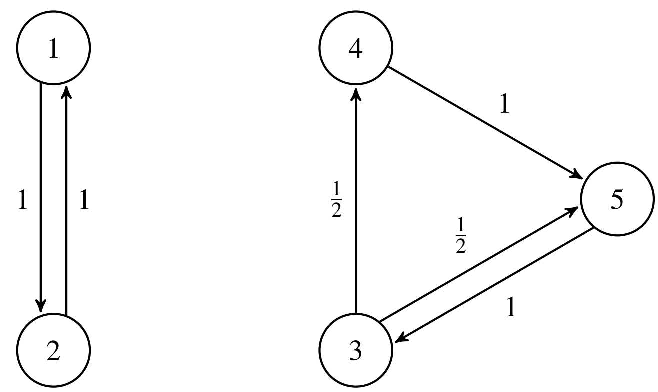
One possible eigenvector associated with eigenvalue 1 is $\begin{bmatrix} -0.61 & -0.31 & -0.23 & -0.69 \end{bmatrix}^T$ (found using IPython). Scaling it by

$$\frac{1}{(-0.61 + (-0.31) + (-0.23) + (-0.69))}$$

so the elements sum to 1, we get $\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}^T$

These are the steady-state frequencies for the pages.

OPTIONAL Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C? Again, graph C with weights in place is shown below. You may use IPython to compute the eigenvalues and eigenvectors again.



Graph C

Solution:

The transition matrix for graph C is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$$

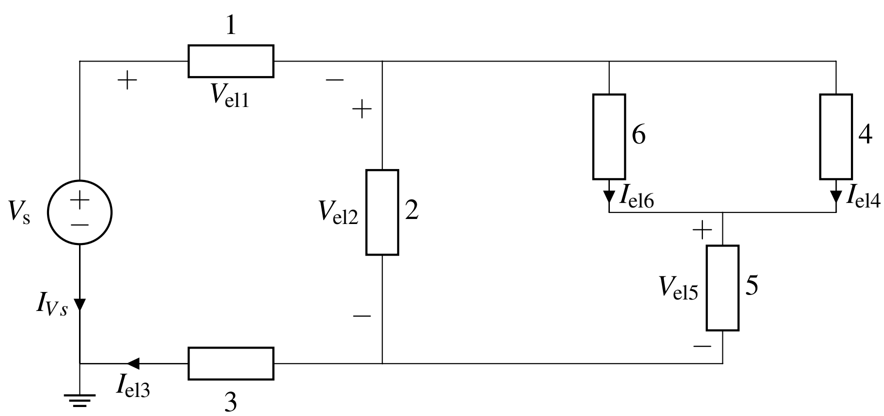
Using IPython, we find that the eigenspace associated with $\lambda = 1$ is spanned by the vectors $\begin{bmatrix} 0 & 0 & 0.4 & 0.2 & 0.4 \end{bmatrix}^T$ and $\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \end{bmatrix}^T$. While any linear combination of these vectors is an eigenvector, these two particular vectors have a nice interpretation.

The first eigenvector describes the steady-state frequencies for the last three webpages, and the second vector describes the steady-state frequencies for the first two webpages. This makes sense since there are essentially "two internets", or two disjoint sets of webpages. Surfers cannot transition between the two, so you cannot assign steady-state frequencies to webpage 1 and webpage 2 relative to the rest. This is why the eigenspace corresponding to the steady-state has dimension 2.

Assuming that each set of steady-state frequencies needs to add to 1, the first assigns steady-state frequencies of 0.4, 0.2, 0.4 to webpage 3, webpage 4, and webpage 5, respectively. The second assigns steady-state frequencies of 0.5 to both webpage 1 and webpage 2.

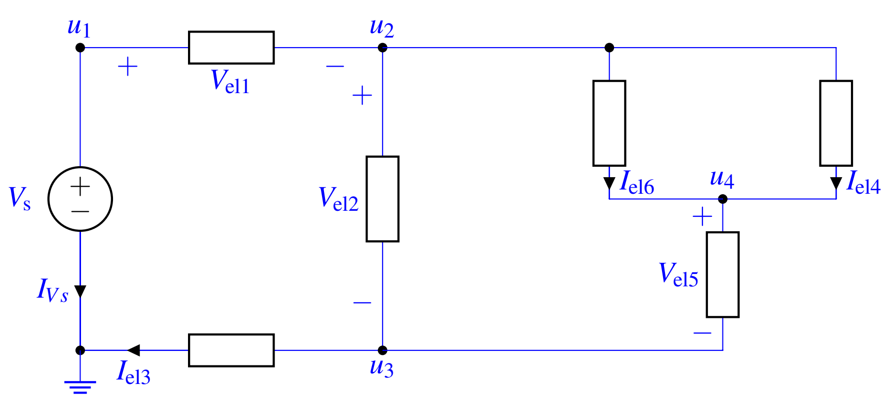
3. Intro to Circuits

Learning Goal: *This problem will help you practice labeling circuit elements and setting up KCL and KVL equations.*



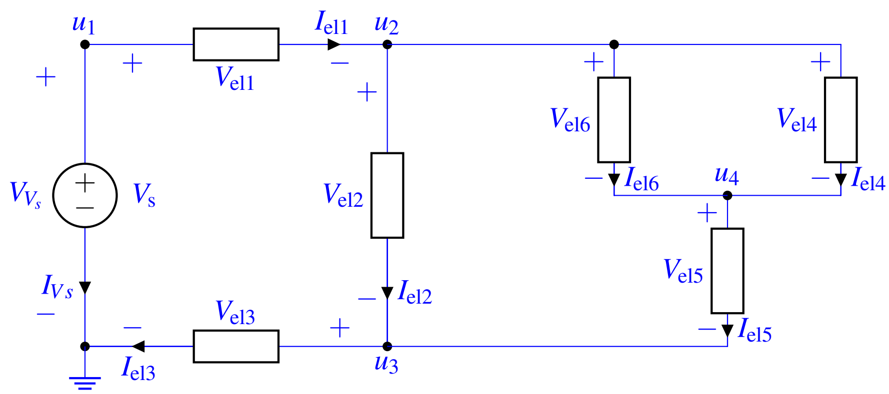
(a) How many nodes does the above circuit have? Label them.
Note: The reference/0V (ground) node has been selected for you, so you don't need to label that, but you need to include it in your node count.

Solution: There are a total of 5 nodes in the circuit, including the reference node. They are labeled u_1 - u_4 below:



(b) Notice that elements 1 - 6 and the voltage source V_s have either the *voltage across* or the *current through* them not labeled. Label the missing *voltages across* or *currents through* for elements 1 - 6, and the voltage source V_s, so that they all follow **passive sign convention**.

Solution: The passive sign convention dictates that the current flows from the positive to the negative terminal of the element (or equivalently exiting the negative terminal / entering the positive terminal if you prefer):



(c) Express all element voltages (including the element voltage across the source, V_s) as a function of node voltages. This will depend on the node labeling you chose in part (a).

Solution: For our specific node labeling we can write:

$$V_{V_s} = u_1 - 0 = u_1 (= V_s)$$

$$V_{el1} = u_1 - u_2$$

$$V_{el2} = u_2 - u_3$$

$$V_{el3} = u_3 - 0 = u_3$$

$$V_{el4} = u_2 - u_4$$

$$V_{el5} = u_4 - u_3$$

$$V_{el6} = u_2 - u_4$$

Notice that the element voltage is always of the form: $V_{el} = u_+ - u_-$.

(d) Write one KCL equation that involves the currents through elements 1 and 2.
*Hint: This will **not** be specific to your node labeling. Your answer may contain currents through other elements too.*

Solution: The only node for which we can write a KCL equation involving both elements 1 and 2 is node u_2, since they only intersect on that node:

$$I_{el1} = I_{el2} + I_{el6} + I_{el4}$$

(e) Write a KVL equation for all the loops that contain the voltage source V_s. These equations should be a function of element voltages and the voltage source V_s.

Solution: Notice that there are in fact 3 loops that contain the voltage source V_s, for which we can write the following equations, starting each time from the reference node and ending at the reference node:

$$V_s - V_{el1} - V_{el2} - V_{el3} = 0$$

$$V_s - V_{el1} - V_{el6} - V_{el5} - V_{el3} = 0$$

$$V_s - V_{el1} - V_{el4} - V_{el5} - V_{el3} = 0$$

The reason this is not specific to our labeling is that the polarity of all elements is either given or set through the passive sign convention.

4. Reverse Eigenvalues

Learning Goal: *Understand how to construct a matrix with a particular set of eigenvalues and eigenvectors.*

In lecture, homework, and section, we have seen a number of ways to compute eigenvalues and eigenvectors from a particular matrix, and explored what they mean in terms of how the matrix transforms vectors. In this problem, we will explore this in the reverse direction by designing it to have a desired set of eigenvalues. Recall the fundamental eigenvector/eigenvalue equation:

$$Av = \lambda v \tag{5}$$

(a) Suppose you are given the following eigenvalue/eigenvector pairs:

$$\lambda_1 = 1, \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1, \vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

These eigenvectors/eigenvalues will result in a reflection matrix about the vector [1, 1]; in other words every component orthogonal to [1, 1] will flipped in sign. Explicitly write out the matrix-vector equations for the two eigenvector/eigenvalue pairs. Make sure to identify each component of the **A** matrix and fill in the relevant values for the eigenvector and eigenvalue. Assume the unkown components of **A** are $a_{11}, a_{12}, a_{21}, a_{22}$

Solution: As suggested by the question, we begin with the eigenvector/eigenvalue equation:

$$Av = \lambda v \tag{6}$$

Let us select the first pair, λ_1 and $\vec{v_1}$, and explicitly write out the components of the **A** matrix in the equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{7}$$

The second pair can be written similarly:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{8}$$

Simplifying:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{9}$$

(b) Reformat the equations you wrote above into a system of linear equations, with $a_{11}, a_{12}, a_{21}, a_{22}$ as unknowns.

Solution: Working out the matrix multiplication yields 4 equations (2 from each set), as follows:

$$1a_{11} + 1a_{12} = 1 \tag{10}$$
$$1a_{21} + 1a_{22} = 1 \tag{11}$$
$$-1a_{11} + 1a_{12} = 1 \tag{12}$$
$$-1a_{21} + 1a_{22} = -1 \tag{13}$$

(c) Now, setup a matrix-vector system of equations and solve for the **A** matrix. Think carefully about what the unknowns in your system are when setting it up.

Solution: The unknowns are the components of the **A** matrix (the a_{ij}s) We can create the matrix-vector formulation as follows:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \tag{14}$$

You can then solve this system however you would like, with the result:

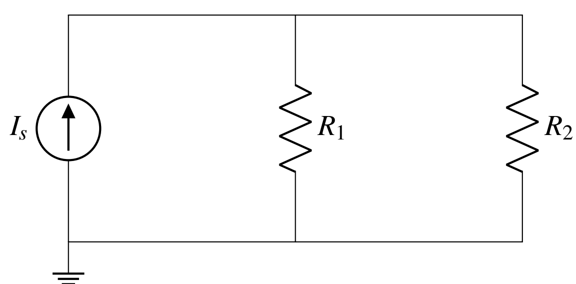
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{15}$$

5. Circuit Analysis

Learning Goal: *This problem will help you practice circuit analysis using the node voltage analysis (NVA) method.*

Using the steps outlined in lecture or in Note 11, analyze the following circuits to calculate the currents through each element and the voltages at each node. Use the ground node labelled for you. You may use a numerical tool such as IPython to solve the final system of linear equations.

(a) $I_s = 3\,\text{mA}, R_1 = 2\,\text{k}\Omega, R_2 = 4\,\text{k}\Omega$



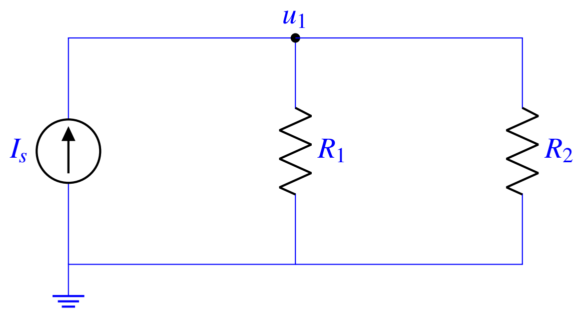
Solution:

Step 1)Pick a junction and label it as u = 0 (“ground”), meaning that we will measure all of the voltages in the rest of the circuit relative to this point.

Step 2) Label all remaining junctions as some “u_i”, representing the voltage at each junction relative to the zero junction/ground.

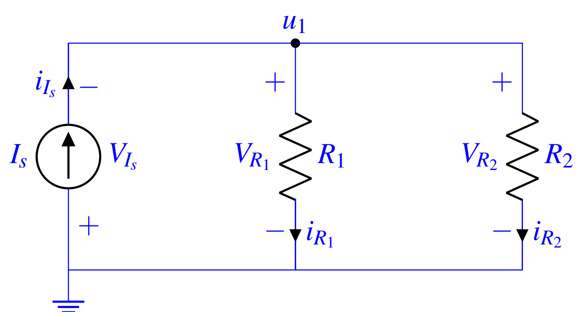
Step 3) Label Remaining Nodes

Label the current through every element in the circuit “in”. Every element in the circuit that was listed above should have a current label, including ideal wires. The direction of the arrow which direction of current flow you are considering to be positive. At this stage of the algorithm, you can pick the direction of all of the current arrows arbitrarily - as long as you are consistent with this choice and follow the rules described in the rest of this algorithm, the math will work out correctly



Step 4) Label Element Voltages and Currents

Next we mark all element voltages and currents.
Start with the current. The direction is arbitrary (top to bottom, bottom to top, it won't matter, but stick with your choice in subsequent steps). Then mark the element voltages following the passive sign convention, i.e. the voltage and current point in the "same" direction.



Step 5) KCL Equations
Write KCL equations for all nodes with unknown voltage, which is only u_1.

$$(-)i_{I_s} + i_1 + i_2 = 0$$

Step 6) Element Currents
Find expressions for all element currents in terms of voltage and element characteristics (e.g. Ohm's law) for all circuit elements except voltage sources. In this circuit there are three, R_1, R_2, and I_s.

$$i_{I_s} = I_s$$
$$i_{R_1} = \frac{V_{R_1}}{R_1}$$
$$i_{R_2} = \frac{V_{R_2}}{R_2}$$

Step 7) Element Voltages

Rewrite the element voltages using the node differences.

$$i_{I_s} = I_s$$
$$i_{R_1} = \frac{u_1}{R_1}$$
$$i_{R_2} = \frac{u_1}{R_2}$$

Step 8) Substitute Element Currents in KCL Equations
Now we substitute the expressions derived in Step 7 into the KCL equations from Step 5.

$$-I_s + \frac{u_1}{R_1} + \frac{u_1}{R_2} = 0$$

We can isolate the unknown terms (u_1) on the left and the known on the right

$$\frac{u_1}{R_1} + \frac{u_1}{R_2} = I_s$$

We only have one equation to solve. Setting up the matrix equation would just be the same as solving this equation. Solving for u_1, we get

$$u_1 = I_s \frac{R_1 R_2}{R_1 + R_2}$$

Plugging in the values we were given, we get

$$u_1 = 3\,\text{mA} \frac{2\,\text{k}\Omega \cdot 4\,\text{k}\Omega}{2\,\text{k}\Omega + 4\,\text{k}\Omega}$$
$$= 4\,\text{V}$$

Node u_1 is $4\,\text{V}$ relative to the ground node we defined. If we had defined the top node as ground, then the bottom node would have measured as $-4\,\text{V}$. Similarly, as drawn, we have $V_{R_1} = V_{R_2} = 4\,\text{V}$; if we flipped the polarities, i.e. swapped + and -, we would have $V_{R_1} = V_{R_2} = -4\,\text{V}$.
The branch currents can be obtained from the node voltages and element equations. Therefore, we can write:

$$i_{R_1} = \frac{u_1}{R_1} = 2\,\text{mA}$$
$$i_{R_2} = \frac{u_1}{R_2} = 1\,\text{mA}$$

Note that

$$i_{R_1} = I_s \frac{R_2}{R_1 + R_2} \qquad i_{R_2} = I_s \frac{R_1}{R_1 + R_2}$$

These are very similar equations to the voltage divider circuit. We call this circuit a current divider.

6. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.