EECS 16A   Designing Information Devices and Systems I
Fall 2023   Midterm 1

Exam Location: Wheeler 150

PRINT your student ID: _____

PRINT AND SIGN your name: _____, _____ _____
                              (last name)   (first name)   (signature)

PRINT your discussion section and GSI: _____

Name and SID of the person to your left: _____

Name and SID of the person to your right: _____

Name and SID of the person in front of you: _____

Name and SID of the person behind you: _____

1. **Honor Code (0 Points)**
   Acknowledge that you have read and agree to the following statement and sign your name below:
   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.
   If you do not sign your name, you will get a 0 on the exam.

2. **When the exam starts, write your SID at the top of every page. (2 Points)**
   No extra time will be given for this task.

3. **Tell us about something you did in the last year that you are proud of. (1 Point)**
   Any answer, as long as you write it down, will be given full credit.

4. **What is a movie you watched recently that made you happy? (1 Point)**
   Any answer, as long as you write it down, will be given full credit.

Do not turn this page until the proctor tells you to do so. You may work on the questions above.
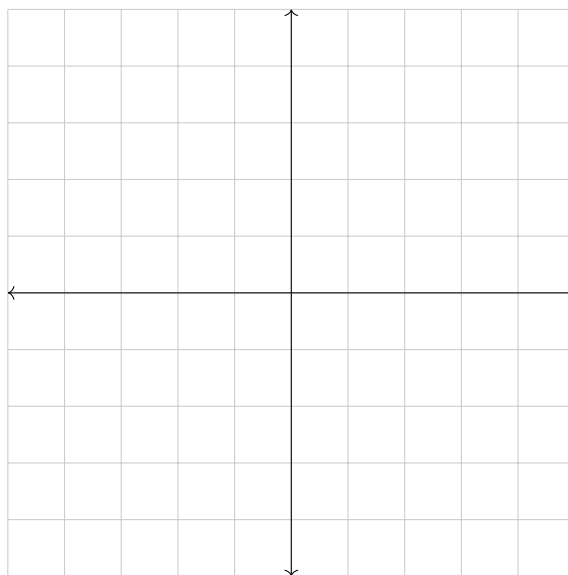
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Extra page for scratchwork.
If you want any work on this page to be graded, please tell us exactly where to look in the problem's solution area.
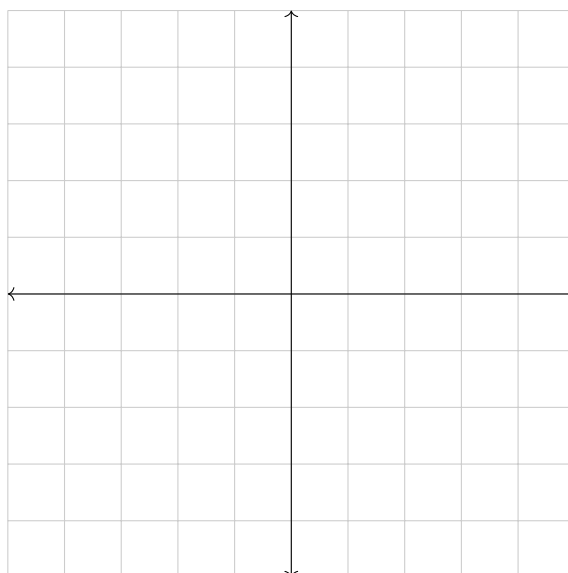
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5. Transform that Vector! (8 points)

(a) (2 points) Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. **Graph the vectors $\vec{x}$ and $\vec{y}$ and label them**.

(b) (3 points) Now apply a transformation matrix $\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ to get new vectors $\vec{v} = \mathbf{T}\vec{x}$ and $\vec{w} = \mathbf{T}\vec{y}$.

Graph the vectors $\vec{v}$ and $\vec{w}$ and label them.

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(c) (3 points) Let $\vec{x} \in \mathbb{R}^2$. You first transform $\vec{x}$ by $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and then by $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. **Write the entries of the matrix $\mathbf{C} \in \mathbb{R}^{2 \times 2}$ that represents the combined transformation**. Show your work.

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6. Shiny New Monitor (12 points)

You are trying to calibrate your new computer monitor. For this you ask the monitor to generate one color for all of its pixels, represented by $\vec{x}_{\text{disp}} = \begin{bmatrix} r \\ g \\ b \end{bmatrix}$, where r, g, b represent the intensities of red, green, and blue light output by the monitor.

You are calibrating using sensors that measure linear combinations of red, green, and blue light. For sensor i, the sensor measurement m_i is

$$m_i = s_{ir}r + s_{ig}g + s_{ib}b \tag{1}$$

where r, g, b are the intensities of red, green, and blue light output by the monitor, and s_{ir}, s_{ig}, s_{ib} are the sensitivities to red, green, and blue respectively for that sensor. The sensitivities of your three sensors are given by:

	Sensitivity to red (s_{ir})	Sensitivity to green (s_{ig})	Sensitivity to blue (s_{ib})
Sensor 1	0.75	0	0.5
Sensor 2	1	1	1
Sensor 3	1	0.5	0

(a) (4 points) You run the calibration and collect the following data:

	Sensor measurement (m_i)
Sensor 1	380
Sensor 2	420
Sensor 3	220

Write a matrix vector equation that allows you to compute $\vec{x}_{\text{disp}}$. You do not have to solve this equation.

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(b) (5 points) Suppose the sensor data for a different measurement gives you the matrix equation:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0.5 & 1 & 1 \\ 0 & 0.5 & 0.5 \end{bmatrix} \vec{x}_{\text{disp}} = \begin{bmatrix} 420 \\ 360 \\ 120 \end{bmatrix}. \tag{2}$$

Use Gaussian elimination to find the vector $\vec{x}_{\text{disp}}$ that satisfies the equation. If there is a unique solution, state it. If there is no solution, explain why. If there are infinite solutions, parameterize your solution. Show your work.

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(c) (3 points) You are interested in developing a new display technology that includes a fourth white channel (w). The white channel can be expressed as a linear combination of all the other channels as shown below:

$$w = \frac{1}{3} \cdot r + \frac{1}{3} \cdot g + \frac{1}{3} \cdot b. \tag{3}$$

Find a transformation matrix T that transforms the vector $\begin{bmatrix} r \\ g \\ b \end{bmatrix}$ **to the vector** $\begin{bmatrix} w \\ r \\ g \\ b \end{bmatrix}$.

$$\mathbf{T} \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} w \\ r \\ g \\ b \end{bmatrix}.$$

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7. Campus WiFi (13 points)

EECS16A is tasked with troubleshooting campus WiFi connectivity issues.

The campus has three WiFi **transmitters**. Let $x_1, x_2, x_3 \in \mathbb{R}$ be the signal strength generated by each transmitter. In addition, we set up three WiFi **detectors** across campus that measure values $d_1, d_2, d_3 \in \mathbb{R}$ respectively.
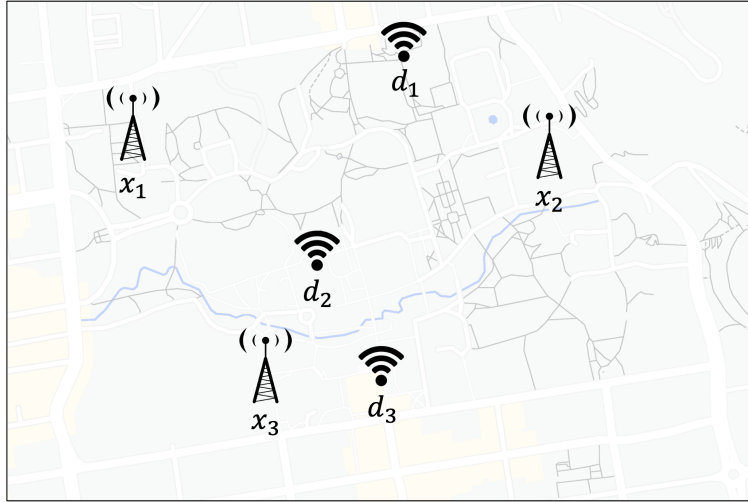
The transmitted signal strengths and measured detector values are represented in a transmitter vector $\vec{x}$ and detector vector $\vec{d}$ as follows

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

After careful study, we have found that $\vec{x}$ and $\vec{d}$ are related by the signal mapping matrix $\mathbf{M}$ as shown below:

$$\mathbf{M}\vec{x} = \vec{d}. \tag{4}$$

(a) (4 points) For this part consider

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

What is the dimension of the column space of M? Justify your answer.

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(b) (4 points) Due to campus budget cuts, we are only allowed to use two detectors. Our transmitter and detector vectors are now

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

For this part we are given a new mapping matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix}.$$

We notice that certain combinations of transmitter signal strengths produce interference that results in no signal at our detectors. We suspect this is connected to the null space. **Find a basis for the null space of M.** Show your work.

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(c) (5 points) For this part consider a new signal mapping matrix $\mathbf{M}$ which has null space

$$\text{null}(\mathbf{M}) = \text{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

We find that we can achieve $\vec{d} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ with the following transmitter vector:

$$\mathbf{M} \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Unfortunately having large values in the transmitter vector $\vec{x}$ make it expensive to operate! UC Berkeley tells us they can only set the signal strength to an integer value from 0 to 5. **Can you find a transmitter vector $\vec{x}$ such that $\mathbf{M}\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ where $x_1, x_2, x_3 \in \{0, 1, 2, 3, 4, 5\}$? If yes, state the vector and show your work. If not justify why no such vector exists.**

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8. Observing Cars (10 points)

Consider an autonomous car with state vector $\vec{x}[t] \in \mathbb{R}^n$ at time t where

$$\vec{x}[t] = \begin{bmatrix} x_1[t] \\ x_2[t] \\ \vdots \\ x_n[t] \end{bmatrix}.$$

$\vec{x}[t]$ updates according to the following equation for every timestep t:

$$\vec{x}[t+1] = \mathbf{A}\vec{x}[t], \mathbf{A} \in \mathbb{R}^{n \times n}. \tag{5}$$

The state of the car is observed through LIDAR measurements $\vec{y}[t] \in \mathbb{R}^m$, which are related to the state of the car for every timestep t as follows:

$$\vec{y}[t] = \mathbf{C}\vec{x}[t], \mathbf{C} \in \mathbb{R}^{m \times n}. \tag{6}$$

In this question, we are concerned with the design of the matrix $\mathbf{C}$.

(a) (3 points) **Express $\vec{y}[t+1]$ in terms of $\vec{x}[t], \mathbf{A},$ and $\mathbf{C}$.** Show your work.

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(b) (7 points) Let $n > 5$. Assume that the first 5 columns of $\mathbf{A}$ are in the null space of $\mathbf{C}$ and no other columns of $\mathbf{A}$ are in the null space of $\mathbf{C}$. **Write $\vec{y}[t+1]$ in terms of $\mathbf{C}$, $x_1[t], \ldots, x_n[t]$, and the columns of A. Which entries of $\vec{x}[t]$ play no role in the value of $\vec{y}[t+1]$?** Justify your answer and show your work.
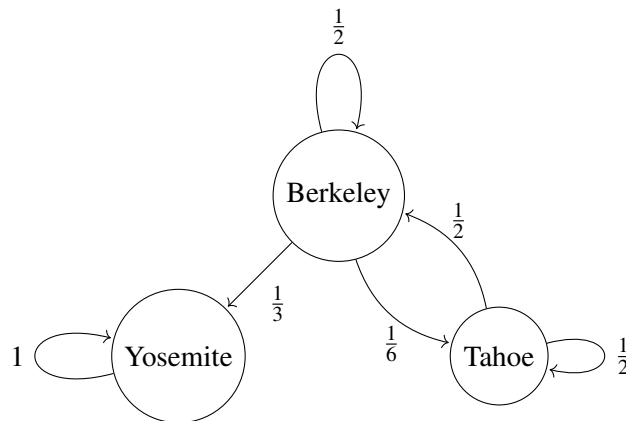
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9. Migration of the Bears (23 points)

A population of black bears migrate between Berkeley, Yosemite, and Tahoe every year. The population of black bears in each location is represented in a state vector $\vec{b}[i]$ defined as

$$\vec{b}[i] = \begin{bmatrix} \text{number of black bears in Berkeley in year } i \\ \text{number of black bears in Yosemite in year } i \\ \text{number of black bears in Tahoe in year } i \end{bmatrix}.$$

The migration of the black bears follows the state transition system modeled by the following diagram:

(a) (4 points) **Find the state transition matrix for this system T such that $\vec{b}[i+1] = \mathbf{T}\vec{b}[i]$ and state whether the system is conservative.** Justify your answer.

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(b) (5 points) Due to climate change, we find that the migration patterns of the bears have changed and the new state transition matrix **S** such that $\vec{b}[i+1] = \mathbf{S}\vec{b}[i]$ is as follows:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{2}{3} & 1 \end{bmatrix}.$$

Find the matrix M such that $\vec{b}[i-1] = \mathbf{M}\vec{b}[i]$**.** Show your work.

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(c) (6 points) Berkeley researchers discover a new species of bear, the Oski bears (Ursus Oskius), which migrate between the three locations according to the transition matrix:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

This system has a steady state. Compute the eigenspace corresponding to the steady state. Show your work.

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(d) (4 points) As part of your undergraduate research project, you are asked to simulate the migration of the Oski bears. Suppose you are given a new transition matrix $\mathbf{Q}$ and an initial state

$$\vec{b}[0] = 108\vec{v}_1 + 36\vec{v}_2,$$

where $\vec{v}_1$ and $\vec{v}_2$ denote the eigenvectors corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{5}{6}$ respectively for the matrix $\mathbf{Q}$. **Find the state vector after two timesteps $\vec{b}[2]$ in terms of $\vec{v}_1$ and $\vec{v}_2$.** Show your work.

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(e) (4 points) We are also interested in the wild fish population in Yosemite and Tahoe, represented by $\vec{a}[i]$, as they are an important source of food for the bears. The fish population changes according to $\vec{a}[i+1] = \mathbf{R}\vec{a}[i]$ where

$$\mathbf{R} = \begin{bmatrix} \frac{2}{5} & \frac{1}{3} \\ \frac{3}{5} & \frac{2}{3} \end{bmatrix}.$$

Compute the eigenvalues of R. Does R have a steady state? Justify your answer and show your work.

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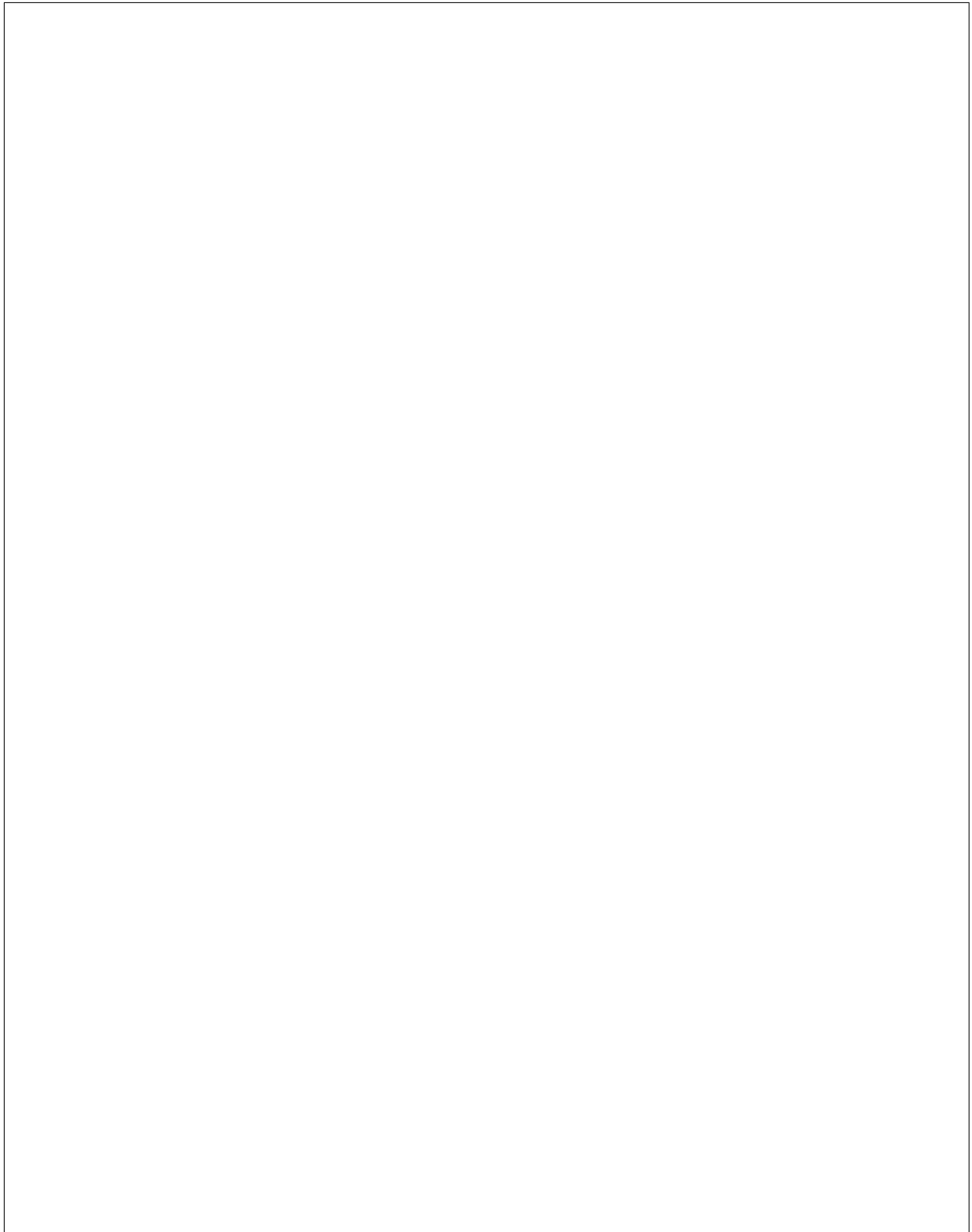
10. Proof (10 points)

Prove the following statement:

If $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a **linearly independent set**, then $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \cdots, \vec{v}_{n-1} + \vec{v}_n, \vec{v}_n\}$ is also a **linearly independent** set.

Hint 1: If $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ are linearly independent, then $\sum_{i=1}^{n} \alpha_i \vec{v}_i = \vec{0}$ if and only if $\alpha_i = 0$ for all i.

Hint 2: Try proving this first for $n = 3$. Partial credit will be awarded for a valid proof for $n = 3$.

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Doodle page!
Draw us something if you want or give us suggestions, compliments, or complaints.
You can also use this page to report anything suspicious that you might have noticed.

Read the following instructions before the exam.

You may fill out the information on the first page before the exam starts. If you are not attending discussion, please write n/a for your discussion GSI name.

There are 10 problems of varying numbers of points. Not all subparts of a question are related to each other. You have 80 minutes for the exam. The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can.

There are 24 pages on the exam, so there should be 12 sheets of paper in the exam. The exam is printed double-sided. Do not forget the problems on the back sides of the pages! Notify a proctor immediately if a page is missing. **Do not tear out or remove any of the pages. Do not remove the exam from the exam room.**

No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.

Write your student ID on each page before time is called. If a page is found without a student ID, we are not responsible for identifying the student who wrote that page.

You may consult ONE handwritten $8.5'' \times 11''$ note sheet (front and back). No phones, calculators, tablets, computers, other electronic devices, or scratch paper are allowed.

Please write your answers legibly in the boxed spaces provided on the exam. The space provided should be adequate. **If you still run out of space, please use a blank page and clearly tell us in the original problem space where to look for your solution.**

Show all of your work in order to receive full credit. Partial credit will be given for substantial progress on each problem.

If you need to use the restrooms during the exam, bring your student ID card, your phone, and your exam to a proctor. You can collect them once you return from the restrooms.

Our advice to you: if you can't solve the problem, state and solve a simpler one that captures at least some of its essence. You might get some partial credit, and more importantly, you will perhaps find yourself on a path to the solution.

Good luck!

Do not turn this page until the proctor tells you to do so.