

**Pledge of Academic Integrity**

By my honor, I affirm that

- (1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;
- (2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;
- (3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and
- (4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

**(10 Points)** In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

*I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.*

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Full Name: \_\_\_\_\_

Signature: \_\_\_\_\_

Date: \_\_\_\_\_

Student ID: \_\_\_\_\_

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**MT3.1 (30 Points) Hide and Seek**

Members of 16A course staff are playing hide and seek in Cory Hall! In an effort to win, they try various different strategies to mask their coordinates.

- (a) (15 points) Melissa tries encrypting her coordinates with the following setup, where her real coordinates are  $\mathbf{x} \in \mathbb{R}^2$  and her encrypted coordinates are  $\mathbf{z} \in \mathbb{R}^2$  by a change of basis. In other words, we can represent the relation between the  $\mathbf{x}$  and  $\mathbf{z}$  as:

$$\mathbf{x} = \mathbf{C}\mathbf{z}$$

Melissa first starts at the origin in the standard basis, with  $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . She first moves one unit north in the standard basis to  $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , and her new encrypted position is  $\mathbf{z}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ . Starting from this new position, she then moves east one unit in the standard basis to  $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , and her final encrypted position is  $\mathbf{z}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

What is the change of basis matrix  $\mathbf{C}$  mapping her coordinates from the encrypted basis to the standard basis?

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- (b) (15 points) Melissa decides to switch her strategy. She encrypts her coordinates with a new unknown change of basis matrix  $\mathbf{T}$ , but shares the eigenvectors for the matrix:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Athul, a seeker, figures out the eigenvalues  $\lambda_1 = -1$  for  $\mathbf{v}_1$  and  $\lambda_2 = 5$  for  $\mathbf{v}_2$ . Given Melissa's encrypted coordinates are  $\mathbf{z}_c = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ , find her real position.

*Hint: Write  $\mathbf{z}_c$  as a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . What happens when you apply  $\mathbf{T}$  to this linear combination?*

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**MT3.2 (45 Points) Ridge Regression**

- (a) (15 points) Consider the matrix vector equation  $\mathbf{Ax} = \mathbf{b}$  where

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solve for  $\hat{\mathbf{x}}$  using least squares.

- (b) (10 points) Now consider a different  $\mathbf{A}$  matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \\ -3 & 6 \end{bmatrix}$$

In two sentences or less, explain why the traditional least squares formula won't work for this matrix.

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It turns out that we can still run least squares with this matrix through a variant called **ridge regression**. Traditional least squares solves the following optimization problem

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|^2$$

Ridge regression adds what's called an *L2 regularization* term to this objective function

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|^2 + \|\eta\mathbf{x}\|^2, \eta > 0 \in \mathbb{R}$$

The resulting ridge regression solution for  $\hat{\mathbf{x}}$  is

$$\hat{\mathbf{x}} = (\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I})^{-1} \mathbf{A}^\top \mathbf{b}$$

In the next couple of parts we will motivate one of the benefits of L2 regularization.

- (c) (10 points) Assume the eigenvalues of  $\mathbf{A}^\top \mathbf{A}$  are  $\lambda_1 = 5$  and  $\lambda_2 = 0$ . Show that the eigenvalues of  $\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I}$  are  $5 + \eta$  and  $\eta$ . *HINT:  $\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I}$  and  $\mathbf{A}^\top \mathbf{A}$  have the same eigenvectors.*

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- (d) (10 points) Suppose that in the general case, we have that the eigenvalues of  $\mathbf{A}^\top \mathbf{A}$  are  $\lambda_1, \lambda_2 \dots \lambda_n$ . Recall that  $\eta$  is strictly greater than 0. Argue that all the eigenvalues of  $\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I}$  must be strictly greater than 0. You may reference the results of the last part even if you did not do the problem. What does this say about the invertibility of  $\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I}$ ?

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### MT3.3 (45 Points) Robot Grasping with PCA

As a robotics automation engineer, we would like to develop a robotic system that can pick up ellipse-shaped objects from a table, using only an overhead camera image as input.

In order to estimate the pose of our elliptical objects, we will characterize it by their center point  $(x, y)$ , while their orientation is described by a unit vector pointing in the direction of their major axis (the longer axis).

To derive these positions, we will sample points in the image that appear to lie on the object. By doing so uniformly across the image, we assume that they are roughly representative of the object's actual position. An example plot is as follows:

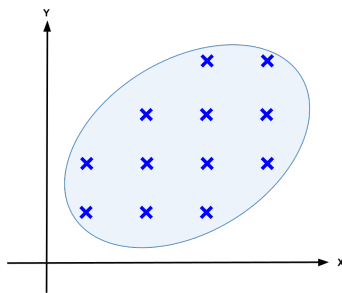


Figure 1: Example of Object Sampling

For one particular sampled object, assume that we store the  $(x, y)$  positions of our sampled points as **row data** in the following matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 1 & 2 \end{bmatrix}$$

- (a) (5 points) Given the data matrix  $\mathbf{A}$ , estimate the position  $(\bar{x}, \bar{y})$  of the center of the ellipse.

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- (b) (15 points) Ignoring the previous subpart, imagine that the data has now been centered about the origin, and can now be characterized by the following data matrix  $\mathbf{D}$ .

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -2 & 2 \\ 2 & -2 \\ -1 & -1 \end{bmatrix}$$

Given this matrix, determine the best estimate for the unit vector  $\hat{\mathbf{m}}$  that characterizes the direction of the ellipse's major axis. This axis should correspond to the **direction of maximum variance** in our sampled points.



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Now that we have successfully estimated the pose of our target object, we can actually pick it up using a gripper as shown below!

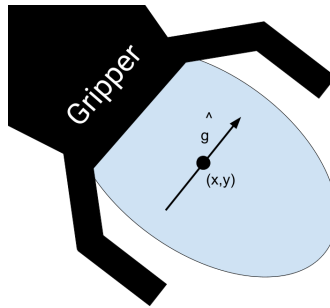


Figure 2: Grasp Visualization

- (c) (5 points) Ignoring the previous subpart, assume we found that  $\hat{\mathbf{m}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$  is the direction of maximum variance. In order to maximize our chances of successfully grasping the object, however, we want to grasp along the **shortest** axis of our ellipse estimation. Provide the unit vector  $\hat{\mathbf{g}}$  along the **shortest** axis.

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Finally, we need to perform *motion planning*, so that we can actually move the gripper to the desired grasp position. For the sake of this problem, let us assume that the gripper's position can be modeled by the following discrete-time system:

$$\mathbf{x}[i + 1] = \mathbf{B}\mathbf{x}[i] + \mathbf{c}u[i]$$

where  $\mathbf{x}[i] = \begin{bmatrix} a \\ b \end{bmatrix}$  is the current position of our gripper at timestep  $i$ ,  $u[i]$  is our scalar control input at timestep  $i$ .

- (d) (10 points) Assume that the gripper begins at state  $\mathbf{x}[0] = \mathbf{0}$ . Express the state of the gripper at timestep 4 as the result of a matrix-vector product of the form:

$$\mathbf{x}[4] = \mathbf{C}\mathbf{u}$$

where  $\mathbf{u} = [u[0] \ u[1] \ u[2] \ u[3]]^\top$  is a vector of the first four control inputs, in terms of  $\mathbf{B}$  and  $\mathbf{c}$ .

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(e) (10 points) Assume that the state of the gripper at timestep 4 can be modeled as:

$$\mathbf{x}[4] = \mathbf{C}\mathbf{u} = \begin{bmatrix} -2 & 1 & 2 & -1 \\ 2 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix}$$

and that our desired grasp position is  $\mathbf{x}^* = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$ . For our application, we need to reach this position in four timesteps, but would like to be as efficient as possible. In particular we would like to find the smallest vector of inputs  $\mathbf{u}^*$ ,  $\mathbf{u}$  where  $\|\mathbf{u}\|$  is *minimized*, such that  $\mathbf{x}^* = \mathbf{C}\mathbf{u}^*$  (we reach our desired position).

Given that the compact SVD of  $\mathbf{C}$  is

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Provide the vector  $\mathbf{u}^*$ .

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### MT3.4 (45 Points) Solving the Damped Harmonic Oscillator

In this problem, we will solve the damped harmonic oscillator, a classical problem in Physics. Suppose you have a spring, whose one side is attached to a wall and the other to a ball of mass  $m$ . Using Newton's 2nd law of motion, the force in a spring, and the viscosity of a fluid in which the spring is in, we can derive a governing differential equation of this system as:

$$\ddot{y}(t) = -\frac{c}{m}\dot{y}(t) - \frac{k}{m}y(t) \quad (1)$$

where  $y(t)$  is the displacement of the mass,  $k$  is the spring constant and  $c$  is the viscous damping coefficient. For simplicity, assume  $m = 1$ , so we get that:

$$\ddot{y}(t) = -c\dot{y}(t) - ky(t) \quad (2)$$

- (a) (10 points) Set up the state space VDE to describe our system using  $\mathbf{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$ .

That is, find the matrix  $\mathbf{A}$  such that

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad (3)$$

- (b) (10 points) Suppose that  $c = 2$ ,  $k = 1$ . Solve for the eigenvalues of the above system. Is the system stable?

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- (c) (15 points) Suppose that  $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$  with  $c = 26$  and  $k = 25$ , we get that  $\mathbf{A}$  can be diagonalized as

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 25 & 1 \end{bmatrix} \begin{bmatrix} -25 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{24} & \frac{1}{24} \\ -\frac{25}{24} & -\frac{1}{24} \end{bmatrix} \quad (4)$$

Given that the system starts at the initial condition  $y(0) = 1$  and  $\dot{y}(0) = 0$ , solve for  $y(t)$ . What is the final position of the mass as  $t$  approaches infinity? In other words, what is  $\lim_{t \rightarrow \infty} y(t)$ ?

*Hint: Recall the specific form in which we want the solution to be found in. How can you determine the final position of the mass using the solution?*

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- (d) (10 points) Now suppose that there is an external force that applies a constant force to the spring. In other words, the new governing equation is

$$\ddot{y}(t) = -cy(t) - ky(t) + F \quad (5)$$

where  $F$  is some constant. Suppose that  $y_h(t)$  is the homogeneous solution. Solve for the general solution of the system in terms of  $y_h(t)$ ,  $F$ ,  $c$ , and  $k$ . You may not need all of these.

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- In some parts of a problem, we may ask you to establish a certain result—for example, “show this” or “prove that.” Even if you’re unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.
- If we ask you to provide a “reasonably simple expression” for something, by default we expect your expression to be in closed form—one *not* involving a sum  $\sum$  or an integral  $\int$ —*unless* we explicitly tell you otherwise.
- Noncompliance with these or other instructions from the teaching staff—including, for example, commencing work prematurely, or continuing it beyond the allocated time window—is a serious violation of the Code of Student Conduct.

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- **(5 Points)** On the *front of every* page, print legibly your last name and ALL digits of your SID. For every page on which you do not write your last name and SID, you forfeit a point, up to the maximum 5 points.
- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, ...*) of the Pledge of Academic Integrity on page 1 of this document. Your solutions will *not* be evaluated without this.
- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand if in-person, or send an email to `eeecs16a@berkeley.edu` if online.
- **This document consists of pages numbered 1 through 18.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.
- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.  
The exam starts at 8:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of 1.5× and 2× time windows have 120 and 160 minutes, respectively.
- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" × 11" sheets of handwritten, original notes having no appendage.  
Collaboration is not permitted.  
Computing, communication, and other electronic devices (except dedicated time-keepers) must be turned off.  
Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper. Additionally, pages 15 and 16 can also be used as scratch paper, **but we will not grade these pages.**
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.