EECS 16A Designing Information Devices and Systems I
Spring 2016 Elad Alon, Babak Ayazifar — Midterm 1

Exam location: 10 Evans, Last name starting with A-B or R-T

PRINT your student ID: ____

PRINT AND SIGN your name: ________________, ________________ ____________________
(last) (first) (signature)

PRINT your Unix account login: ee16a-_____________

PRINT your discussion section and GSI (the one you attend): ____

Name of the person to your left: ____

Name of the person to your right: ____

Name of the person in front of you: ____

Name of the person behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What other courses are you taking this term? (1 pt)**

2. **What activity do you really enjoy? Describe how it makes you feel. (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

Section 1: Straightforward questions (24 points)

Unless told otherwise, you must show work to get credit. There will be very little partial credit given in this section. Each problem is worth 8 points.

3. Mechanical Johann

Invert the following matrix:

$$\begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 4 \\ 2 & 4 & 10 \end{bmatrix}$$

Solutions: We can find the inverse by row reducing the augmented system:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 2 & 4 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 2 & 4 & 10 & 0 & 0 & 1 \end{array}\right] \underbrace{\Longrightarrow}_{R_3 \leftarrow R_3 + (-2) \cdot R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 & 1 \end{array}\right] \underbrace{\Longrightarrow}_{R_3 \leftarrow R_3 \nabla \cdot (-2)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1/2 \end{array}\right] \underbrace{\Longrightarrow}_{R_2 \leftarrow R_2 + (-4) \cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1/2 \end{array}\right] \underbrace{\Longrightarrow}_{R_1 \leftarrow R_1 + (-6) \cdot R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -5 & 0 & 3 \\ 0 & 1 & 0 & -4 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1/2 \end{array}\right] \underbrace{\Longrightarrow}_{R_1 \leftarrow R_1 + (-2) \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -1 \\ 0 & 1 & 0 & -4 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1/2 \end{array}\right]$$

Thus the inverse is given by:

$$\begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & 2 \\ 1 & 0 & -1/2 \end{bmatrix}$$

We can check our answer by calculating

$$\begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 4 \\ 2 & 4 & 10 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & 2 \\ 1 & 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Freedom!

Let A and B be $n \times n$ matrices. Suppose A is invertible, but B is not. **Prove that Rank(AB) $<$ Rank(A).**

(That is, show that the number of linearly independent columns in AB is strictly less than the number of linearly independent columns in A).

Solutions: **Proof 1:**
Since $\mathbf{B}$ is rank deficient, it has linearly dependent columns. This means that there exists a nonzero vector $\vec{x}$ such that

$$\mathbf{B}\vec{x} = \vec{0}.$$

Premultiplying this equation by the matrix $\mathbf{A}$ gives

$$\mathbf{A}(\mathbf{B}\vec{x}) = \mathbf{A}\vec{0} = \vec{0}.$$

But

$$\mathbf{A}(\mathbf{B}\vec{x}) = (\mathbf{AB})\vec{x},$$

so we've shown that

$$(\mathbf{AB})\vec{x} = \vec{0}$$

for some nonzero vector $\vec{x}$. This proves that the columns of the $n \times n$ matrix $\mathbf{AB}$ are linearly dependent, which in turn means that rank$(\mathbf{AB}) < n$. **Proof 2:**
Define the product matrix $\mathbf{C} = \mathbf{AB}$. Writing the matrix $\mathbf{B}$ in column form, we find that

$$\begin{aligned}
\mathbf{C} &= \mathbf{AB} \\
&= \mathbf{A}\left[\vec{b}_1 \cdots \vec{b}_\ell \cdots \vec{b}_n\right] \\
&= \left[\mathbf{A}\vec{b}_1 \cdots \mathbf{A}\vec{b}_\ell \cdots \mathbf{A}\vec{b}_n\right] \\
&= \left[\vec{c}_1 \cdots \vec{c}_\ell \cdots \vec{c}_n\right],
\end{aligned}$$

where $\vec{c}_\ell = \mathbf{A}\vec{b}_\ell$ denotes the ℓ^{th} column of the matrix $\mathbf{C} = \mathbf{AB}$.

Since $\mathbf{B}$ is rank deficient its columns must be linearly dependent. This means that we can write any column of $\mathbf{B}$ as a nontrivial linear combination of the others (by nontrivial we mean that not all the coefficients in the linear combination are zero). For example, let's write the last column of $\mathbf{B}$ as a linear combination of the first $n-1$ columns. That is,

$$\vec{b}_n = \sum_{\ell=1}^{n-1} \alpha_\ell \vec{b}_\ell,$$

for some set of coefficients $\alpha_1, \ldots, \alpha_{n-1}$, *not all zero*.

Now let's look at the last column of the product matrix $\mathbf{C} = \mathbf{AB}$:

$$\begin{aligned}\vec{c}_n &= \mathbf{A}\vec{b}_n \\ &= \mathbf{A}\sum_{\ell=1}^{n-1}\alpha_\ell \vec{b}_\ell \\ &= \sum_{\ell=1}^{n-1}\alpha_\ell \underbrace{\mathbf{A}\vec{b}_\ell}_{\vec{c}_\ell} \\ &= \sum_{\ell=1}^{n-1}\alpha_\ell \vec{c}_\ell,\end{aligned}$$

which shows that the n^{th} column of $\mathbf{C} = \mathbf{AB}$ is a nontrivial linear combination of the first $n-1$ columns. This means that $\mathbf{C}$ has linearly dependent columns, which in turn implies that $\mathbf{AB}$ is rank deficient.

5. True or False?

You only need to write True or False under each subpart.

(a) There exists an invertible $n \times n$ matrix A for which $A^2 = 0$.

Solutions: False

Let's left multiply and right multiply A^2 by A^{-1} so we have $A^{-1}AAA^{-1}$. By associativity of matrix multiplication, we have $(A^{-1}A)(AA^{-1}) = I_n I_n = I_n$ where I is the identity matrix. However, if A^2 were 0, then $(A^{-1}A)(AA^{-1}) = A^{-1}A^2A^{-1} = 0$ where 0 is a matrix of all zeros, hence resulting in a contradiction.

(b) If A is an invertible $n \times n$ matrix, then for all vectors $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ has a unique solution.

Solutions: True

If A is invertible, then there is a unique matrix A^{-1}. Left multiply the equation by A^{-1}, and we will have $A^{-1}A\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}$, where $\vec{x}$ is a unique vector.

(c) If A and B are invertible $n \times n$ matrices, then the product AB is invertible.

Solutions: True

$(AB)^{-1} = B^{-1}A^{-1}$.
Note that $ABB^{-1}A^{-1} = I$ and $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$

(d) The two vectors $v_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $v_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ form a basis for the subspace $\text{Span}(\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\})$.

Solutions: True.

$\text{Span}(\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\})$ spans the x-y plane in $\mathbb{R}^3$. Since $v_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $v_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ are linearly independent, they form a basis for the x-y plane in $\mathbb{R}^3$ as well.

(e) A set of n linearly dependent vectors in $\mathbb{R}^n$ can span $\mathbb{R}^n$.

Solutions: False

A set of n linearly dependent vectors span some subspace of dimension $0 < \dim(A) < n$ in $\mathbb{R}^n$.
Note: It is incorrect to say the set of linearly dependent vectors spans $\mathbb{R}^{n-1}$ for two reasons. First, you don't know what the dimension is of the subspace it spans, which could be less than $n-1$. Second, there is no such thing as $\mathbb{R}^{n-1} \in \mathbb{R}^n$. The vectors are "in" $\mathbb{R}^n$ based on how many elements are in the vector, and a set of vectors spans some subspace (potentially the entire space.)

(f) For all matrices A and B, where A is 5×5 and B is 4×4, it is always the case that $\text{Rank}(A) > \text{Rank}(B)$.
Solutions: False

Size does not determine rank! For example, if A was a matrix of all ones rank(A) would be 1. If, on the other hand B was an identity matrix it would have full rank: rank(B) = 4.
You can only claim larger size implies larger rank if you assume the matrices are full rank (pivots in every column, all column vectors are linearly independent from the rest.)

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[If you are want the work on this page be graded, please state CLEARLY which problem(s) this space is for.]

Section 2: Free-form Problems (100 points)

6. Faerie Battles (20 points)

In a war of civilizations, the light dragons are fighting for survival against the invading evil imp empire. It is the night before the deciding battle, and Ela Dalon the Human Hobo has proposed using faerie lights to determine enemy numbers. There are four enemy camps.

The faerie lights begin completely dark and get brighter directly proportional to the number of imps present around them, i.e. if Camp X has n imps, any faerie light that visits Camp X will gain n brightness units.

Unfortunately, faerie lights are notoriously difficult to control, and they are so small that the dragons can only measure the total brightness of groups of faerie lights. The Human Hobo therefore proposes to send four groups of faerie lights, and is fortunately able to keep track of how many faerie lights he sent and which camps these faeries actually visited. This results in the following:

Group 1 gained 7 total brightness units

- 1 faerie light visited Camp A
- 5 faerie lights visited Camp B
- 1 faerie light visited Camp C

Group 2 gained 26 total brightness units

- 2 faerie lights visited Camp B
- 4 faerie lights visited Camp C
- 6 faerie lights visited Camp D

Group 3 gained 10 total brightness unints

- 2 faerie lights visited Camp C
- 2 faerie lights visited Camp D

Group 4 gained 9 total brightness units

- 3 faerie lights visited Camp D

(a) Write the system of equations that relates the number of imps in each camp to the total measured brightness units from each group.

Solutions:

$$\begin{array}{ccccccccl} A & + & 5B & + & C & & & = & 7 \\ & & 2B & + & 4C & + & 6D & = & 26 \\ & & & & 2C & + & 2D & = & 10 \\ & & & & & & 3D & = & 9 \end{array}$$

(b) From the given information, can you determine how many imps are at each campsite? If so, report their numbers to help the faeries! If not, explain why not.

Solutions: The matrix is upper triangular, so the matrix's columns must be linearly independent and there is enough information to determine exactly how many enemy imps there are at each campsite.

We can find the number of imps at each campsite by row reducing the following augmented system:

$$\begin{bmatrix} 1 & 5 & 1 & 0 & \vdots & 7 \\ 0 & 2 & 4 & 6 & \vdots & 26 \\ 0 & 0 & 2 & 2 & \vdots & 10 \\ 0 & 0 & 0 & 3 & \vdots & 9 \end{bmatrix} \underbrace{\Longrightarrow}_{\substack{R_2 \leftarrow R_2/2 \\ R_3 \leftarrow R_3/2 \\ R_4 \leftarrow R_4/3}} \begin{bmatrix} 1 & 5 & 1 & 0 & \vdots & 7 \\ 0 & 1 & 2 & 3 & \vdots & 13 \\ 0 & 0 & 1 & 1 & \vdots & 5 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \underbrace{\Longrightarrow}_{R_3 \leftarrow R_3 + (-1) \cdot R_4}$$

$$\begin{bmatrix} 1 & 5 & 1 & 0 & \vdots & 7 \\ 0 & 1 & 2 & 3 & \vdots & 13 \\ 0 & 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \underbrace{\Longrightarrow}_{R_2 \leftarrow R_2 + (-3) \cdot R_4} \begin{bmatrix} 1 & 5 & 1 & 0 & \vdots & 7 \\ 0 & 1 & 2 & 0 & \vdots & 4 \\ 0 & 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \underbrace{\Longrightarrow}_{R_2 \leftarrow R_2 + (-2) \cdot R_3}$$

$$\begin{bmatrix} 1 & 5 & 1 & 0 & \vdots & 7 \\ 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \underbrace{\Longrightarrow}_{R_1 \leftarrow R_1 + (-1) \cdot R_3} \begin{bmatrix} 1 & 5 & 0 & 0 & \vdots & 5 \\ 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \underbrace{\Longrightarrow}_{R_1 \leftarrow R_1 + (-5) \cdot R_2}$$

Thus, we discover that:

- Camp A has 5 imps
- Camp B has 0 imps
- Camp C has 2 imps
- Camp D has 3 imps

We can check our answer by substituting our answers into the original equations:

$$\begin{aligned} 1 \cdot 5 + 5 \cdot 0 + 1 \cdot 2 = 5 + 2 = 7 \\ 2 \cdot 0 + 4 \cdot 2 + 6 \cdot 3 = 8 + 18 = 26 \\ 2 \cdot 2 + 2 \cdot 3 = 4 + 6 = 10 \\ 3 \cdot 3 = 9 \end{aligned}$$

(c) Oh no! You just found out that the evil imps manipulated the faerie lights from Group 1, rendering their information useless. You send in an emergency group of faeries to make up for the loss. They gained a total of 10 brightness units, and visited camps as follows:

- 1 faerie light visited Camp B
- 2 faerie lights visited Camp C
- 3 faerie lights visited Camp D

From this information (together with the information from Groups 2, 3, 4), can you determine how many imps are at each campsite? If so, report their numbers to help the faeries! If not, explain why not.

Solutions: No you cannot determine how many imps are at each campsite.

- The new team does not pass through Camp A so you have no way of determining the number of imps in Camp A.
- The new measurement conflicts with an existing measurement. The system of equations is inconsistent, so there is no solution.
- The new measurement is a linearly dependent one. Specifically, it sends half the faerie lights that the second measurement sends. Therefore, we do not have enough linearly independent measurements to solve for the imps in each camp.

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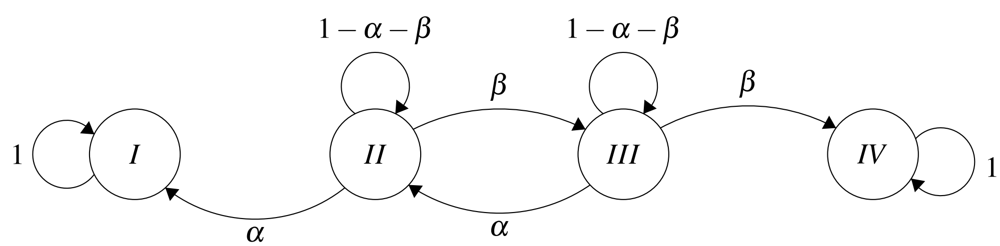
7. A Tale of a Million Technocrats and the Four Dream Cities (24 Points)

This problem is a tale of one million Technocrats and four Dream Cities. Listed in order from West to East, the four cities are (I) San Francisco, (II) Denver, (III) Chicago, and (IV) New York City.

The Technocrats don't die. They don't reproduce. In other words, their total population size is a constant from the initial time $n = 0$ to the end of time as $n \to \infty$. At the strike of each second on a Universal Clock, each Technocrat chooses to either remain at the city he or she is already in, or move instantaneously to another of the four Dream Cities, according to the following rules:

- Every Technocrat who moves to either San Francisco or New York City will stay in that city forever (no more moving thereafter!);
- Whether eastward or westward, every Technocrat moves in single hops—so for example, no direct move from San Francisco to either Chicago or New York City is allowed;
- At each second, the fraction of Technocrats in each city who move eastward legally (e.g. from Denver to Chicago, or from Chicago to New York City) is β;
- At each second, the fraction of Technocrats in each city who move westward legally (e.g. from Chicago to Denver, or from Denver to San Francisco) is α;
- The following inequality must be true: $\alpha + \beta \leq 1$.

The following diagram summarizes these restrictions graphically.



Let the state vector for this system be at time n be:

$$\vec{s}[n] = \begin{bmatrix} s_1[n] \\ s_2[n] \\ s_3[n] \\ s_4[n] \end{bmatrix},$$

where $s_\ell[n]$ denotes number of Technocrats in City ℓ at time n; for example, $s_3[n]$ denotes the Technocrat population in Chicago at time n.

The state-evolution equation for this system is given by

$$\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n],$$

where $\mathbf{A}$ is the state-transition matrix.

(a) Determine the state-transition matrix $\mathbf{A}$.

Solutions:

$$\begin{bmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1-\alpha-\beta & \alpha & 0 \\ 0 & \beta & 1-\alpha-\beta & 0 \\ 0 & 0 & \beta & 1 \end{bmatrix},$$

(b) In this part, we're interested in backward inference of the state vector $\vec{s}[n]$ from a future state vector, say $\vec{s}[n+1]$. As you know, this is possible only if a matrix $\mathbf{A}^{-1}$ exists, so that

$$\vec{s}[n] = \mathbf{A}^{-1}\vec{s}[n+1].$$

Consider a model of the Technocrat migration system described above with $\beta = \alpha = 1/3$. Assume $\vec{s}[1] = \vec{1}$, where $\vec{1}$ denotes the vector of all ones.

$$\vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

For this model of the system described above, either explain why time-reversed inference of the state vector is not possible, or determine $\vec{s}[0]$ from $\vec{s}[1]$ explicitly by computing $\mathbf{A}^{-1}$ first.

Solutions: Model 1:

$$A_1 = \begin{bmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1 \end{bmatrix}.$$

Rows 2 and 3 are identical, so A_1 is not invertible. In other words, the row vectors are linearly dependent and Gaussian Elimination would result in a row of 0's, which can't be inverted.

Model 2:

$$A_2 = \begin{bmatrix} 1 & 1/4 & 0 & 0 \\ 0 & 1/2 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/4 & 1 \end{bmatrix}.$$

This is full rank, so it's invertible. In other words, there is a pivot in every row; every row vector is linearly independent from the rest.

$$A_2^{-1} = \begin{bmatrix} 1 & -2/3 & 1/3 & 0 \\ 0 & 8/3 & -4/3 & 0 \\ 0 & -4/3 & 8/3 & 0 \\ 0 & 1/3 & -2/3 & 1 \end{bmatrix}.$$

$$\vec{s}[0] = A_2^{-1}\vec{1} = \begin{bmatrix} 2/3 \\ 4/3 \\ 4/3 \\ 2/3 \end{bmatrix}.$$

For the remainder of the problem, assume $\beta = \alpha = 1/4$ and that the state transition matrix A is:

$$\mathbf{A} = \begin{bmatrix} 1 & 1/4 & 0 & 0 \\ 0 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1/4 & 1 \end{bmatrix}.$$

Recall that the following equation governs $\vec{s}[n]$, the state of the system at time n, and the initial state $\vec{s}[0]$:

$$\vec{s}[n] = \mathbf{A}^n \vec{s}[0].$$

For each of the remaining parts, assume that a limiting state vector exists, given by

$$\lim_{n\to\infty} \vec{s}[n] = \vec{s}_\infty.$$

(c) Suppose the initial state of the system is given by the vector

$$\vec{s}[0] = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

This means that, initially, 500,000 Technocrats are in San Francisco, and the other 500,000 are in New York City. Without complicated mathematical derivations, determine the state vector $\vec{s}[n]$ for all $n \geq 1$.

Solutions:

$$\vec{s}_\infty = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

With the given state transition diagram, no technocrats would move so $\vec{s}_0 = \vec{s}_t = \vec{s}_\infty$.

(d) Now suppose the initial state of the system is given by the vector

$$\vec{s}[0] = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

This means that, initially, 500,000 Technocrats are in Denver, and the other 500,000 are in Chicago. Without complicated mathematical derivations, determine the limiting state vector $\lim_{n\to\infty} \vec{s}[n]$.

Solutions:

$$\vec{s}_\infty = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

By symmetry, half go to SF and half go to NYC.

(e) Finally, now suppose the initial state of the system is given by the vector

$$\vec{s}[0] = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

This means that, initially, the Technocrats are equally distributed among the four Dream Cities. Without complicated mathematical derivations, determine $\lim_{n\to\infty} \vec{s}[n]$.

Solutions:

$$\vec{s}_\infty = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

By symmetry, half go to SF and half go to NYC. In addition, we can think of the state vector in this part as follows:

$$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

Where the first vector is from part (c) and the second vector is from part(d). So in the limit $n \to \infty$, we get the same distribution as before.

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8. Ayy - Reflections on SIXTEEN (28 points)

You are involved in the design of a robot called SIXTEEN-AYY that has arms with a certain range of motion and certain operations that the arms can perform.

For parts (a)–(e) of this problem, we will look at a simplified case where we pretend that the robot's arm can move only in a 2-dimensional space. Thus, we will represent the position of the end of the robot's arm with a vector $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$. Let's further assume that the robot will be built to perform two movement commands determined by the following transformations:

T_1: Reflects $\vec{p}$ about the line $y = -x$.

T_2: Rotates $\vec{p}$ clockwise about the origin by 45°

(a) Write the matrix A_1 that applies the transformation T_1.

Solutions:

$$\vec{a}_1 = T_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$\vec{a}_2 = T_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

(b) Write the matrix A_2 that applies the transformation T_2.

Solutions:

$$\theta = \frac{-\pi}{4} = -45^\circ$$
$$A_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(c) Leah commands SIXTEEN-AYY to perform T_1 followed by T_2 - find the matrix A_{12} that captures the effect of this sequence of commands.

Solutions:

$$A_{12} = A_2 A_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(d) Being the contrarian that he is, Bob instead wants to command SIXTEEN-AYY to perform T_2 followed by T_1. Find the matrix A_{21} that captures the effect of this sequence of commands.

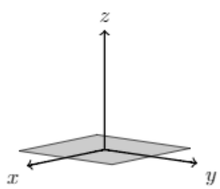
Solutions:

$$A_{21} = A_1 A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

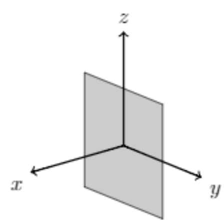
(e) If SIXTEEN-AYY started out in the same initial configuration for both Bob and Leah, will its arm end up in the same place after executing Leah's command as it does after executing Bob's command? Briefly justify why or why not.

Solutions: No, the resulting matrices won't be the same because we've just shown $A_1 A_2 \neq A_2 A_1$; i.e. matrix multiplication is not commutative.

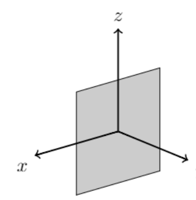
(f) Now let's go back to 3-dimensional space, with the position of the end of the robot arm represented as vectors $\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \in \mathbb{R}^3$. Suppose the end of the robot's arm can reach only points in the span of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Circle the graph below that corresponds to this possible range of motion.



Motion 1: xy plane



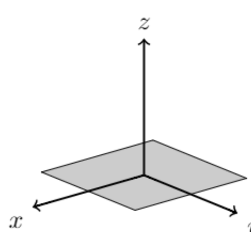
Motion 2: yz plane



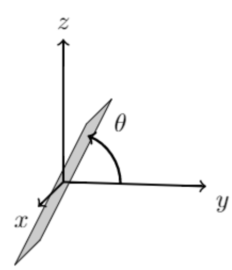
Motion 3: xz plane

Solutions: The span of the vectors is defined by $\begin{bmatrix} P_x \\ P_y \\ 0 \end{bmatrix}$, which is just the xy plane. Thus, our answer is diagram 1.

(g) Finally, we'd like to allow the robot to place the end of its arm at a point that is θ radians above the x-y plane, but as shown below, is still restricted to a two-dimensional plane at that angle. Write the basis vectors that would correspond to this possible range of motion for a fixed θ.
(*Hint*: If you had a point along the y- axis and wanted to rotate it by an angle of θ radians relative to the y-axis while remaining on the y-z plane, what transformation would you apply?)



Motion 4: Before, with no angle



Motion 5: The plane rotated by θ

Solutions: We notice that a rotation relative to the y-axis maintains the same x-value while rotating our y and z by θ radians. Thus, we can write the transformation matrix as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Note that since we only had two basis vectors to begin with, the new range of motion also only needs two basis vectors corresponding to the rotation of the x and y unit vectors, so our basis vectors are:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} \right\}$$

In fact, all rotation matrices in n dimensions can be decomposed into a series of rotations in 2 dimensions. So all rotations are products of the basic rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

generalized to a larger $n \times n$ matrix with 1's in the dimensions that aren't being rotated. Read more on **Wikipedia!**

9. Goodness Gracious, Great Balls of Fur! (28 points)

You are a biologist who has discovered a new species of rodents that live in tunnels underground, and you have named the species "furballs". You have been observing a particular network of tunnels. Figure 1 shows the network of tunnels with three chambers in it. To document their behavior, you observe the number of furballs in each chamber at every minute. After observing the furballs' behavior for a while, you have figured out that their behavior follows a regular pattern. Each minute, a well defined fraction of the furballs in a given chamber move to the other chambers. The fractions you have observed are shown in Figure 1. The fractions of furballs leaving the Play Room could not be determined through your observations, and are shown as the variables p_1, p_2, and p_3

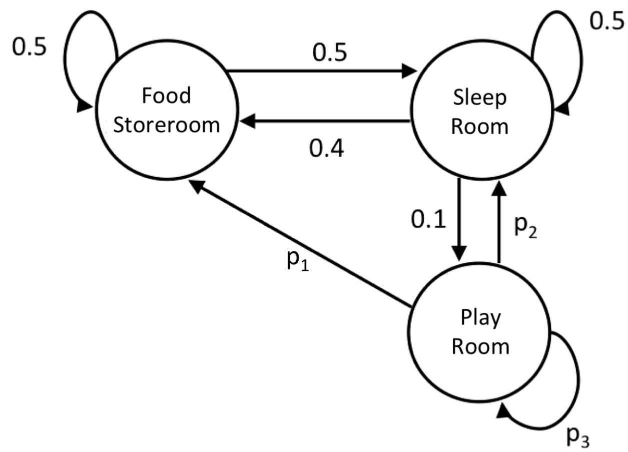


Figure 1

(a) Let the number of furballs in the Food Storeroom at time n be $x_f[n]$, the number of furballs in Sleep room at time n be $x_s[n]$, and the number of furballs in the Play Room at time n be $x_p[n]$. We would like to find the transition matrix A such that, $\begin{bmatrix} x_f[n+1] \\ x_s[n+1] \\ x_p[n+1] \end{bmatrix} = A \begin{bmatrix} x_f[n] \\ x_s[n] \\ x_p[n] \end{bmatrix}$. Write A using the numbers and the variables in the diagram.

Solutions: Write a matrix in which each column corresponds to the outgoing arrows in the diagram from a particular room, and each row corresponds to the incoming arrows to a particular room:

$$\begin{bmatrix} 0.5 & 0.4 & p_1 \\ 0.5 & 0.5 & p_2 \\ 0 & 0.1 & p_3 \end{bmatrix}$$

(b) We know that no furballs enter or leave the configuration of tunnels shown above and that during the time you're observing the behavior, and no furballs die or are born. What constraint does this place on the values of p_1, p_2, p_3? Write your answer in equation form.

Solutions: If number is conserved, this means each column vector in the matrix sums to 1. Therefore:

$p_1 + p_2 + p_3 = 1$

(c) Suppose we let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, $\vec{x}[n+1] = \begin{bmatrix} x_f[n+1] \\ x_s[n+1] \\ x_p[n+1] \end{bmatrix}$ and $\vec{x}[n] = \begin{bmatrix} x_f[n] \\ x_s[n] \\ x_p[n] \end{bmatrix}$, and that we are sure that $x_p[n]$ is nonzero. Express $\vec{p}$ as a function of the numbers in the diagram, $\vec{x}[n]$, and $\vec{x}[n+1]$. (*Hint*: what is the relationship between $\vec{x}[n+1]$ and $\vec{x}[n]$)

Solutions:

$$\begin{bmatrix} 0.5 & 0.4 & p_1 \\ 0.5 & 0.5 & p_2 \\ 0 & 0.1 & p_3 \end{bmatrix} \begin{bmatrix} x_f[n] \\ x_s[n] \\ x_p[n] \end{bmatrix} = \vec{x}[n+1]$$

$$x_f[n] \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} + x_s[n] \begin{bmatrix} 0.4 \\ 0.5 \\ 0.1 \end{bmatrix} + x_p[n] \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \vec{x}[n+1]$$

$$x_p[n] \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \vec{x}[n+1] - x_f[n] \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} - x_s[n] \begin{bmatrix} 0.4 \\ 0.5 \\ 0.1 \end{bmatrix}$$

$$\vec{p} = \frac{1}{x_p[n]} \left(\vec{x}[n+1] - x_f[n] \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} - x_s[n] \begin{bmatrix} 0.4 \\ 0.5 \\ 0.1 \end{bmatrix} \right)$$

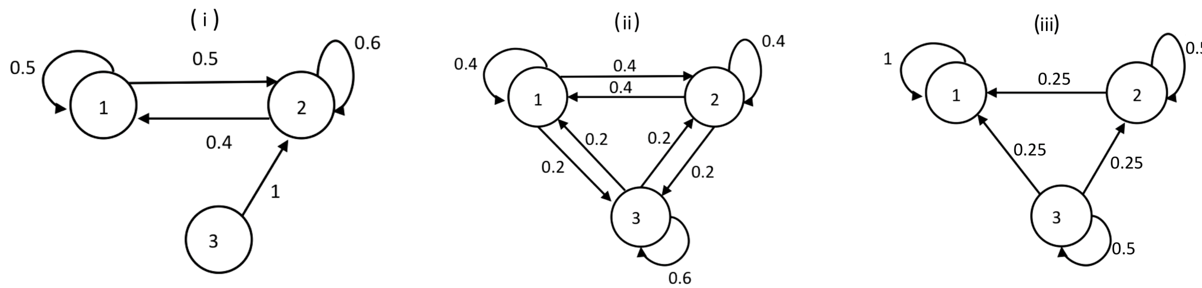
(d) Using part (c), solve for $\vec{p}$ given that $\vec{x}[n+1] = \begin{bmatrix} 22 \\ 30 \\ 48 \end{bmatrix}$ and $\vec{x}[n] = \begin{bmatrix} 20 \\ 30 \\ 50 \end{bmatrix}$.

Solutions:

$$\vec{p} = \frac{1}{50} \left(\begin{bmatrix} 22 \\ 30 \\ 48 \end{bmatrix} - 20 \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} - 30 \begin{bmatrix} 0.4 \\ 0.5 \\ 0.1 \end{bmatrix} \right)$$

$$\vec{p} = \begin{bmatrix} 0 \\ 0.1 \\ 0.9 \end{bmatrix}$$

(e) You discover a new system of tunnels where furballs live. You know this system has 3 chambers like the last one did, but you do not know how the chambers are connected, or the behavior of this colony of furballs. However, if $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix}$ where $x_i[n]$ represents the number of furballs in chamber i at timestep n, where time starts at $n = 0$, you observe that $\vec{x}[2] = \begin{bmatrix} 60 \\ 24 \\ 16 \end{bmatrix}$. Which of the following pump diagrams in Figure 2 represent a possible set of behaviors for this colony?



$$A_i = \begin{bmatrix} 0.5 & 0.4 & 0 \\ 0.5 & 0.6 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_{ii} = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \qquad A_{iii} = \begin{bmatrix} 1 & 0.25 & 0.25 \\ 0 & 0.5 & 0.25 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Figure 2

Solutions:

(i) The third row in A_i consists of all zeros. This means, at any positive timestep, $\vec{x}$ would be forced to have zero as its third entry. Therefore, the given $\vec{x}[2]$ is not in the range of this matrix, and this option do not represent a possible set of behaviors.

(ii) The first two rows in A_{ii} are the same. This means, at any positive timestep, $\vec{x}$ would be forced to have the same number in its first and second entry. Therefore, the given $\vec{x}[2]$ is not in the range of this matrix, and this option do not represent a possible set of behaviors.

(iii) The given $\vec{x}[2]$ is in the range of A_{iii}. Therefore, full credit was given for choosing this option. However, number is not conserved in this option. Since it was ambiguous whether or not number was meant to be conserved in the question, partial credit was given for excluding this option with this explanation.

PRINT your name and student ID: ______________________________

[If you are want the work on this page be graded, please state CLEARLY which problem(s) this space is for. You can also draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]