EECS 16A Designing Information Devices and Systems I
Spring 2023 Midterm 1

Midterm 1 Solution

1. HONOR CODE
Please read the following statements of the honor code, and sign your name (you don't need to copy it).

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my unlimited printed resources.*
- *I did not collaborate with any other human being on this exam.*

2. The Warmup (14 points)

(a) (6 points) In each of the following parts you are given an $m \times n$ matrix $\mathbf{A}$ and an $l \times p$ matrix $\mathbf{B}$. For each of the following subparts:

1) State the dimensions of $\mathbf{A}$ and $\mathbf{B}$. That is, provide values for m, n, l, and p

2) State whether it is possible to multiply $\mathbf{A}$ by $\mathbf{B}$ (i.e., is $\mathbf{AB}$ valid?). If so, write down the resulting product $\mathbf{AB}$.

(i).

$$\mathbf{A} = \begin{bmatrix} 0 & 4 \\ 3 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 3 & 3 \\ 2 & 3 \end{bmatrix}$$

Solution: $m = 2, n = 2, l = 3, p = 2$.

Is $\mathbf{AB}$ valid? If so, compute it.

Solution: $\mathbf{AB}$ is NOT valid because $n \neq l$.

(ii).

$$\mathbf{A} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

Solution: $m = 2, n = 1, l = 1, p = 2$.

Is $\mathbf{AB}$ valid? If so, compute it.

Solution: $\mathbf{AB}$ is valid because $n = l$.

$$\mathbf{AB} = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$$

(b) (2 points) For each of the following matrices, write the transpose.

(i).

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 & 6 \\ 3 & 2 & 2 & 8 \end{bmatrix}$$

Solution:

$$\mathbf{A}^T = \begin{bmatrix} 2 & 3 \\ 4 & 2 \\ 3 & 2 \\ 6 & 8 \end{bmatrix}$$

(ii).

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

$$\mathbf{B}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(c) (6 points) For each of the following functions f, state whether the function is linear or not linear. If it is linear, explicitly show that it satisfies superposition and homogeneity. If it is not linear, show that it fails to satisfy either superposition or homogeneity.

(i). $f(x) = 2(x+2)$

Solution: This function is *NOT* linear. In fact, it is affine.

- Superposition (FAILS)
  $f(x_1 + x_2) = 2(x_1 + x_2 + 2)$.
  But $f(x_1) + f(x_2) = 2(x_1 + 2) + 2(x_2 + 2) = 2(x_1 + x_2 + 4)$.
  Thus, $f(x_1 + x_2) \neq f(x_1) + f(x_2)$
- Homogeneity (FAILS)
  $f(ax) = 2(ax+2)$.
  But $af(x) = 2a(x+2) = 2(ax+2a)$
  Thus, $f(ax) \neq af(x)$

It is only necessary to show that one of the above fails.

(ii). $f(x) = 2x + x$

Solution:

This functions is linear.

We can first simplify the function $f(x) = 2x + x = 3x$

- Superposition (PASSES)
  $f(x_1 + x_2) = 3(x_1 + x_2) = 3x_1 + 3x_2 = f(x_1) + f(x_2)$
- Homogeneity (PASSES)
  $f(ax) = 3ax = a3x = af(x)$.
  Both need to be shown for full credit.

3. Oski at the Store (14 points)

Oski works at the Cal Student Store and is trying to figure out how heavy various items are. Unfortunately, he cannot measure each item individually. Instead, Oski can only weigh boxes that have a mixture of different items within them. Specifically, Oski has 3 boxes:

- The first box has 1 hoodie, 1 mug, and 2 textbooks; it weighs 8 lbs.
- The second box has 2 hoodies, 3 mugs, and 4 textbooks; it weighs 17 lbs.
- The third box has 4 hoodies, 5 mugs, and 8 textbooks; it weighs 33 lbs.

Oski is interested in knowing the individual weights of hoodies (h), mugs (m), and textbooks (t).

(a) (6 points)

(i). Formulate the problem as a matrix-vector equation $\mathbf{A}\vec{x} = \vec{b}$, where each entry of $\vec{x}$ corresponds to the individual weight of an item (i.e., hoodie h, mug m, and textbook t). That is, fill out the entries for $\mathbf{A}$ and $\vec{b}$ below such that $\mathbf{A}\vec{x} = \vec{b}$.

Solution: We can write the following system of equations:

$$h + m + 2t = 8$$
$$2h + 3m + 4t = 17$$
$$4h + 5m + 8t = 33$$

Writing this as a matrix-vector equation, we get:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 8 \end{bmatrix} \vec{x} = \begin{bmatrix} 8 \\ 17 \\ 33 \end{bmatrix}$$

(ii). Use Gaussian elimination to solve the system. If there are no solutions, say so. If there are infinite solutions, write the general form. Can Oski uniquely determine the individual weight of every item?

Solution:

Our augmented matrix looks like

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 2 & 3 & 4 & 17 \\ 4 & 5 & 8 & 33 \end{array}\right]$$

We run the operations $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - 4R_1$ to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Now, we run the operation $R_3 \leftarrow R_3 - R_2$ to get our row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, we backsubstitute by running $R_1 \leftarrow R_1 - R_2$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We can see that the third variable t is a free variable. Let t be equal to some parameter $c \in \mathbb{R}$. Then, our general form looks like:

$$\begin{bmatrix} 7-2c \\ 1 \\ c \end{bmatrix}$$

No, while Oski can figure out that a mug weighs 1 pound as given by the solution, he cannot figure out the weights of a hoodie or textbooks because there are infinitely many possible solutions to these.

Note, the additional explanation above is not required, only the Gaussian elimination and the statement "No".

(b) (4 points) Oski's friend Nathan is using a *different* set of boxes to find the weights of hoodies, mugs and textbooks. He construct the following augmented matrix to solve for $\vec{x} = \begin{bmatrix} h \\ m \\ t \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 3 & 4 & 6 & 20 \end{array}\right]$$

Select the correct statement about Nathan's system. Justify your response in the space below using Gaussian elimination.

○ The system has one unique solution.

○ The system has infinite solutions.

● The system has no solutions.

Solution: There are no solutions to this system because the equations are inconsistent.

Proceeding with Gaussian Elimination:

We run the operations $R_3 \leftarrow R_3 - 3R_1$ which gives:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & -1 \end{array}\right]$$

Next, we run the operations $R_3 \leftarrow R_3 - 4R_2$ which gives:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

But if we look at the last row, we see that we have an inconsistent equation since $0+0+0 \neq -5$.

(c) (4 points) Now, completely separate from the previous parts, Oski is trying to measure the weights of coats (c) and teddy bears (t) using the same strategy. This time he has two boxes:

- The first box has 1 coat and 3 teddy bears; it weighs 5 lbs.
- The second box has 5 coats and 1 teddy bear; it weighs 11 lbs.

This system is represented by the matrix-vector equation $\mathbf{A}\vec{x} = \vec{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix} \qquad \vec{\mathbf{x}} = \begin{bmatrix} c \\ t \end{bmatrix} \qquad \vec{\mathbf{b}} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

Find the inverse of $\mathbf{A}$ and use it to solve for $\vec{x}$.

Solution:

Writing this as a matrix-vector equation, we get:

$$\begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

We can use the formula for the inverse of a 2 x 2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Applying this formula, we find that

$$A^{-1} = \frac{1}{1 * 1 - 3 * 5} \begin{bmatrix} 1 & -3 \\ -5 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{-1}{14} & \frac{3}{14} \\ \frac{5}{14} & \frac{-1}{14} \end{bmatrix}$$

Now, to solve for $\vec{x}$, because $A\vec{x} = \vec{b}$, we can compute $\vec{x} = A^{-1}\vec{b}$.

$$\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = \begin{bmatrix} \frac{-1}{14} & \frac{3}{14} \\ \frac{5}{14} & \frac{-1}{14} \end{bmatrix} \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

4. Matrix Spaces (10 points)

Consider the following $n \times m$ matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \\ | & | & & | \end{bmatrix}$$

where $\vec{a}_1, \ldots, \vec{a}_m \in \mathbb{R}^n$ represent the columns of $\mathbf{A}$ and $m \geq n$.

(a) (2 points) What are the minimum and maximum possible values for the dimension of the **column space** of $\mathbf{A}$?

Solution: minimum = 0, maximum = n

(b) (2 points) Select all of the *True* statements below.

Solution: *Note: The first option below is actually true for $m > n$ and false for $m = n$. For that reason we accepted either answer.*

- ☐ The dimension of the column space is **maximum** when $\vec{a}_1, \ldots, \vec{a}_m$ are linearly *dependent*.
- ■ The dimension of the column space is **maximum** when $\vec{a}_1, \ldots, \vec{a}_m$ spans $\mathbb{R}^n$.
- ■ The dimension of the column space is **minimum** when $\vec{a}_1 = \ldots = \vec{a}_m = \vec{0}$.
- ☐ The dimension of the column space is **minimum** when $\vec{a}_1, \ldots, \vec{a}_m$ are linearly *independent*.

(c) (2 points) What are the minimum and maximum possible values for the dimension of the **null space** of $\mathbf{A}$?

Solution: minimum = $m - n$, maximum = m

(d) (2 points) Select all of the *True* statements below.

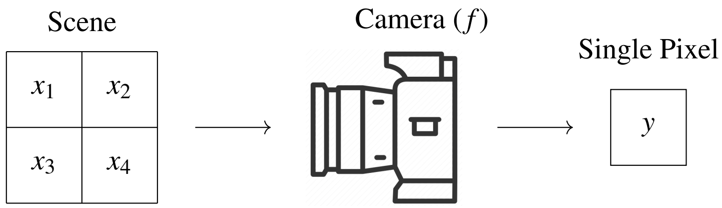
- ■ The dimension of the null space is **maximum** when $\mathbf{A}\vec{x} = \vec{0}$ for every $\vec{x} \in \mathbb{R}^m$.
- ☐ The dimension of the null space is **maximum** when $\vec{a}_1, \ldots, \vec{a}_m$ form a basis for $\mathbb{R}^n$.
- ■ The dimension of the null space is **minimum** when the dimension of the column space is maximum.
- ☐ The dimension of the null space is **minimum** when the dimension of the column space is minimum.

(e) (2 points) Select the conditions which *must* be true if $\vec{a}_1, \ldots, \vec{a}_m$ is a basis of $\mathbb{R}^n$.

- [x] $m = n$
- [ ] $m > n$
- [ ] $\vec{a}_1, \ldots, \vec{a}_m$ are linearly *dependent*
- [x] $\vec{a}_1, \ldots, \vec{a}_m$ spans $\mathbb{R}^n$
- [x] $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution for *any* $\vec{b} \in \mathbb{R}^n$.

5. Single-pixel Camera (12 points)

Consider imaging a scene with a single pixel camera, as shown below.

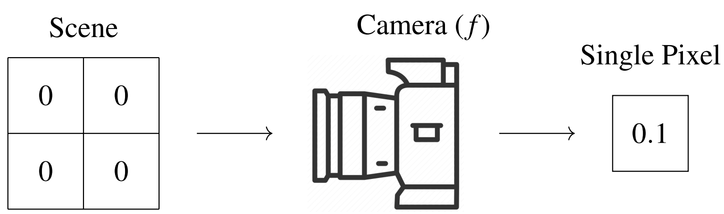


We can model the scene as a 2x2 grid of light intensity values, (x_1, x_2, x_3, x_4), and the single pixel image with y. They are related by a function f, which represents the effects of the camera, such that:

$$y = f(x_1, x_2, x_3, x_4)$$

.

(a) (3 points) Suppose you take the camera into a pitch black room (with no light), such that the scene is all 0s (i.e., $x_1 = x_2 = x_3 = x_4 = 0$). When you take an image of the scene, you a pixel value of $y = 0.1$.



(i). Could f be linear?

○ Yes

● No

(ii). If yes, provide an example of a linear f such that $y = f(x_1, x_2, x_3, x_4)$. If no, provide a brief justification for why not.

Solution: No f cannot be linear. If it were linear, f could be represented as a linear combination of the inputs x_1, x_2, x_3, x_4 and in the given case that means there would exist a linear combination of 0s equal to 0.1, which is not possible.

(b) (3 points) Now suppose you are given a new camera and move to a new area that has light. The function representing this new camera is g, which means $y = g(x_1, x_2, x_3, x_4)$. You are told that g **is a linear function**, which means it can be written as a linear combination of its inputs (i.e., $g = ax_1 + bx_2 + cx_3 + dx_4$).

How many different scenes would you need image in order to solve for the constants a, b, c, d? Assume there is no redundancy between different scenes.

Note that you do not need to actually solve for anything, just tell us how many different scenes we would need image in order to completely specify g.

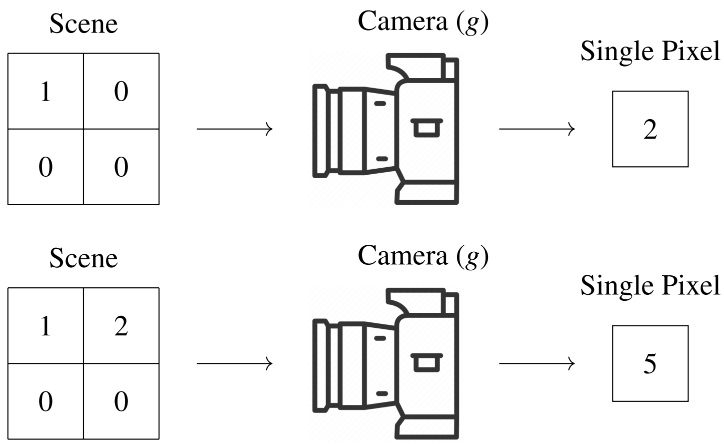
Solution: 4 Scenes

Since g is linear, we can write it as $y = g(x_1, x_2, x_3, x_4) = ax_1 + bx_2 + cx_3 + dx_4$. Thus there are 4 unknowns which requires 4 linearly independent equations. We can obtain these equations by capturing 4 different scenes.

The above explanation is not required.

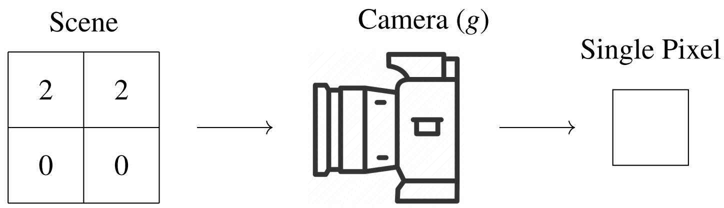
(c) (4 points)

Now suppose you use this new camera, represented by the linear function g, to take two images, each of a different scene shown below. You are given the output pixel value y for each of these scenes.



For the following scenes, fill in the corresponding output value y in the box labeled Single Pixel. Show your work.

i. **Scene 1:**



Solution:
The scene in this case is just a sum of the two given scenes. Thus by linearity, it's image should be the sum of the images of the two scenes. $y = 2 + 5 = 7$.

Alternatively, we can use the given measurements to first solve for the coefficients a and b of the linear function g and then use them to find y for the new scenes. To do this, we write out the given scenes as a system of equations:

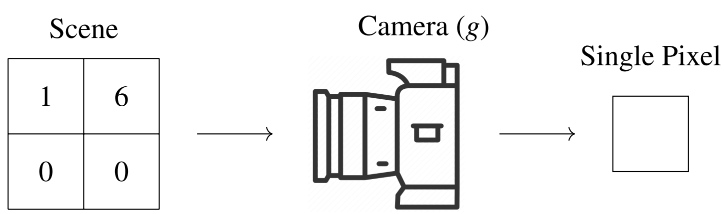
$$a = 2,\ a + 2b = 5$$

. Solving gives that $a = 2$ and $b = \frac{3}{2}$. Now we can compute our new scene as

$$2a + 2b = 2(2) + 2(\frac{3}{2}) = 7.$$

which matches our first method.

ii. **Scene 2:**



Solution: The scene in this case is 3 times the second scene minus 2 time s the first scene. Thus by linearity $y = 3 \times 5 - 2 \times 2 = 11$.

Just like the previous part, we can also use the coefficients a and b:

$$a + 6b = 2 + 9 = 11.$$

which matches the first method.

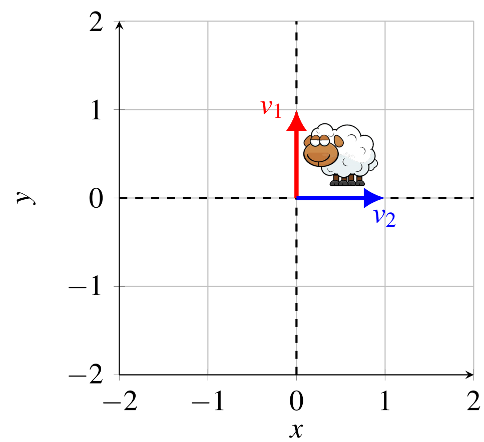
(d) (2 points) Why were you able to solve for the missing y in part c) despite having only measured 2 scenes?

Solution: This is because the bottom row (x_3 and x_4) in all of the given scenes is only 0s. Thus it does not matter what their coefficients c and d are for the linear equation. This effectively reduces the problem to solving only for two coefficients, a and b, which are associated with x_1 and x_2 respectively. Then, since we have two scenes, we have enough information to solve for those coefficients and solve for y.

6. Shearing Sheep (14 points)

In the following questions we will consider transforming an image with a matrix. Recall that an image can be thought of as a collection of points v_i. When we transform an image with a matrix $\mathbf{A}$, we are mapping each v_i to a corresponding $u_i = \mathbf{A}v_i$. This new collection of points, u_i, comprises the transformed image. For each image below, we highlight two out of many points in its collection with the vectors labeled v_1 and v_2.

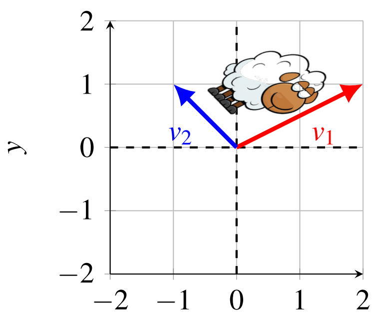
(a) (4 points) First consider the image shown below and the matrix transformation $\mathbf{A} = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$



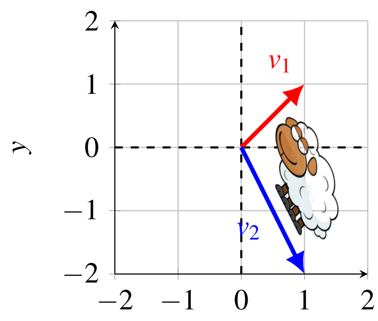
Recall that $\vec{v}_1$ and $\vec{v}_2$ lie on the image. In this case, $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(i). Which of the following represents the result of transforming the given image with the matrix $\mathbf{A}$? Fill in the bubble for either Option 1, Option 2, or Option 3.

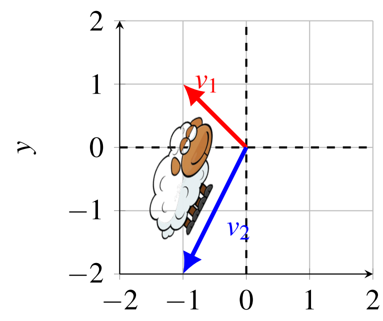
○ **Option 1.**



○ **Option 2.**



● **Option 3.**



Solution:

Option 3. is the correct choice.

Transforming a specific vector $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ yields

$$\mathbf{A}\vec{v}_2 = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

which is only compatible with image (3).

(ii). Is the above transformation reversible? Justify your answer mathematically.

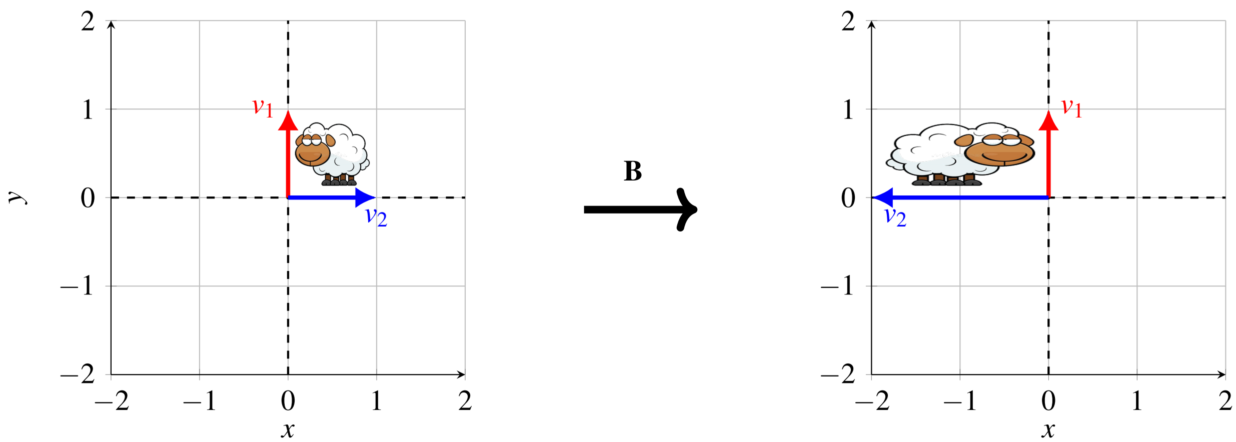
Solution:

Yes, the transformation is reversible because the matrix A is invertible. Thus to reverse the transformation we would simply apply A^{-1}.

The matrix **A** is invertible since it has a nonzero determinant/it's square with linearly independent columns.

(b) (4 points)

Derive the matrix **B** which results in the below transformation.



Solution:

To solve this problem, pick two pixels $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ on the image and relate the coordinates before and after the transformation.

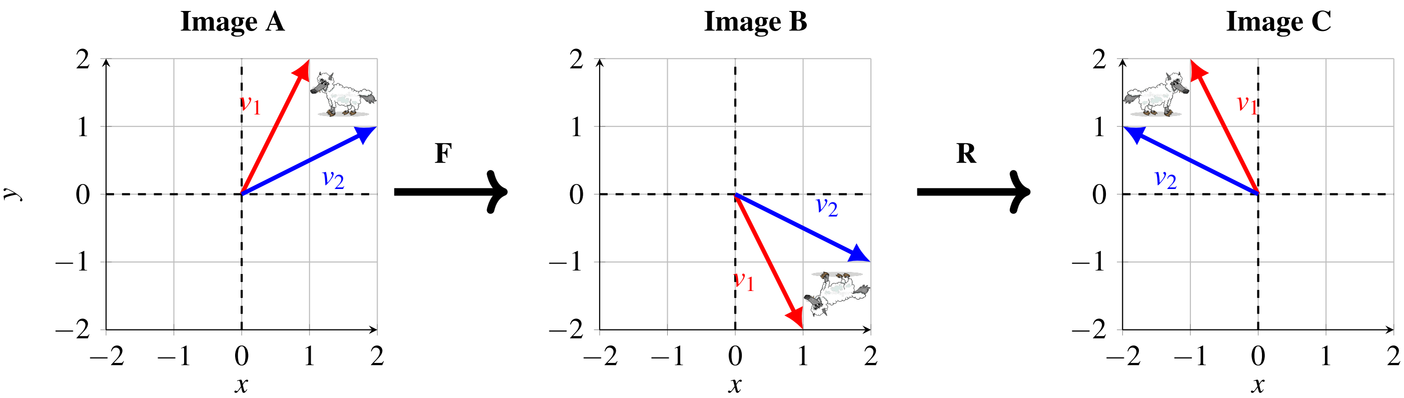
For unit vector $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ the transformed vector is $\vec{b}_1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

For unit vector $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the transformed vector is $\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

From inspection the transformation matrix **B**, it appears to scale the x coordinate by -2 and the y coordinate by 1. Thus, the transformation matrix is

$$\mathbf{B} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) (6 points) Now suppose we perform a sequence of transformations on a new image (shown below as Image A). Beginning Image A, we first apply a transformation **F** to get the middle image, Image B, and then apply another transformation with matrix **R** to get the rightmost image, Image C.



Specifically, $\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects the image about the x-axis and $\mathbf{R} = \begin{bmatrix} \cos(180^\circ) & -\sin(180^\circ) \\ \sin(180^\circ) & \cos(180^\circ) \end{bmatrix}$ rotates the image counter-clockwise by 180°:

(i). Let $\vec{v}$ be a point on Image A and let $\vec{d}$ be the corresponding transformed point on Image C (after undergoing both transformations). Write d in terms of $\vec{v}$, $\mathbf{F}$, and $\mathbf{R}$.
Solution: The Image B point will be $\mathbf{F}\vec{v}$ and so the Image C point will be $\mathbf{RF}\vec{v}$.

(ii). Consider an arbitrary sequence of matrix transformations (not the F and R shown above). *In general*, will the order of transformations affect the final transformed image? That is, does it matter which matrix I apply first?

- ● Yes, the order of transformations will change the final image.
- ○ No, the order of transformations will *not* change the final image.

Explain your answer using matrix properties.

Solution: Yes, in general, the order of transformations matter because matrices do not generally commute. That is, $\mathbf{AB} \neq \mathbf{BA}$ for matrices $\mathbf{A}$ and $\mathbf{B}$.

(iii). Now consider the *specific* transformations F and R given above in the problem statement. Will switching the order of these transformations (i.e., applying rotation R first, then reflection F) result in the same final transformed image (Image C)?

- ○ Yes, the order of F and R changes the final image.
- ● No, the order of F and R does *not* change the final image.

Justify your response mathematically.

Solution:

The order in this specific case does *NOT* matter. The matrix $\mathbf{F}$ and $\mathbf{R}$ are commutative for the angle $\theta = 180°$. In this case,

$$\mathbf{R} = \begin{bmatrix} \cos(180°) & -\sin(180°) \\ \sin(180°) & \cos(180°) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

thus,

$$\mathbf{R} \cdot \mathbf{F} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

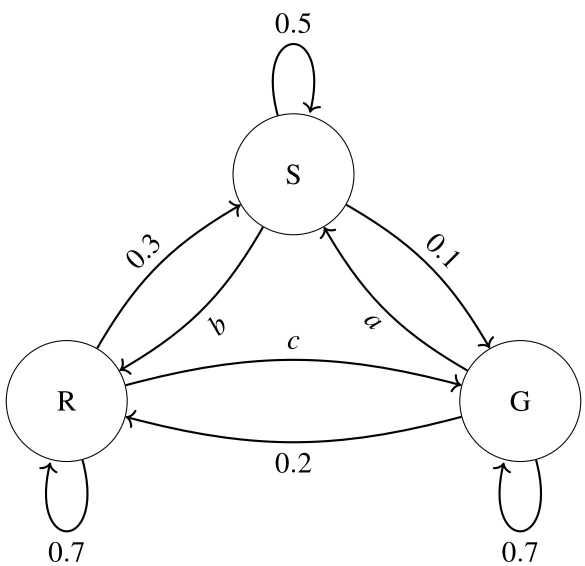
$$\mathbf{F} \cdot \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

which are equivalent. Thus the final image transformation will be equivalent.

7. Water Transitions (16 points)

With all the recent (and desperately needed) rain in California, Aniruddh is interested in modeling the flow of water between different forms of water storage: snowpack (S), reservoir (R), and groundwater (G). Below is a diagram showing his state transition model for the change in water distribution.

Note that some of the transition values are symbolic (e.g., a, b, c)



(a) (4 points)

Let $\vec{m}[t] = \begin{bmatrix} m_s[t] \\ m_r[t] \\ m_g[t] \end{bmatrix}$ where $m_s[t], m_r[t], m_g[t]$ represent the volume of water in S (snowpack), R (reservoir), and G (groundwater) at time t.

(i). Fill in the below matrix $\mathbf{A}$ such that $\vec{m}[t+1] = \mathbf{A}\vec{m}[t]$. Your answer may contain a, b, and c.

(ii). What values of a, b and c would make this system conservative?

Solution: Reading out the diagram,

$$\mathbf{A} = \begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.3 & a \\ b & 0.7 & 0.2 \\ 0.1 & c & 0.7 \end{bmatrix}$$

For the system to be conservative:

$a + 0.2 + 0.7 = 1 \implies a = 0.1$

$0.5 + b + 0.1 = 1 \implies b = 0.4$

$0.3 + 0.7 + c = 1 \implies c = 0.0$

(b) (6 points) Anish also takes a look at the problem, and comes up with a different model and directly provides you with the following state transition matrix. $\mathbf{B} = \begin{bmatrix} 4/5 & 1/5 & 3/10 \\ 1/10 & 7/10 & 1/10 \\ 1/10 & 1/10 & 3/5 \end{bmatrix}$

(i). Does a steady-state exist for this model?

● Yes

○ No

(ii). If so, define the set of steady-state vectors. If not, why not?

Solution: A steady state vector is a vector, $\vec{v}$ which satisfies:

$$\mathbf{B}\vec{v} = \vec{v} \tag{1}$$

In other words, this is the eigenvector of B associated with eigenvalue 1. The most efficient way to answer this question is to assume that B has eigenvalue 1 and solve for its eigenvector. If no solution exists, this means there is no steady state. If a solution exists, then we will arrive at it. We begin by noting that the eigenvector will satisfy:

$$\mathbf{B}\vec{v} - \vec{v} = \mathbf{B}\vec{v} - \mathbb{I}\vec{v} = (\mathbf{B} - \mathbb{I})\vec{v} = 0 \tag{2}$$

In other words, it lies in the nullspace of $\mathbf{B} - \mathbf{I}$. We can perform straightforward Gaussian elimination to compute the set of solutions to this matrix. Our augmented matrix looks like

$$\left[\begin{array}{ccc|c} -1/5 & 1/5 & 3/10 & 0 \\ 1/10 & -3/10 & 1/10 & 0 \\ 1/10 & 1/10 & -2/5 & 0 \end{array}\right]$$

We solve the Gaussian Elimination to yield:

$$\left[\begin{array}{ccc|c} 1 & 0 & -11/4 & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We can see that we will have infinite solutions, with v_3 a free variable. Thus, we can say that the set of all solutions is:

$$\left\{ \begin{pmatrix} (11/4)v_3 \\ (5/4)v_3 \\ v_3 \end{pmatrix} | v_3 \in \mathbb{R} \right\} \tag{3}$$

(c) (6 points) Finally, Vivian also comes up with a model of her own. However, she does not directly give you a state transition matrix. Instead, she tells you the following information:

- The eigenvalues of her transition matrix, **C** are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = \frac{1}{2}$.
- The eigenvectors corresponding to these eigenvalues are:
$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix},$

Suppose you know the current state of the system (at time t) is $\vec{m}[t] = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}$. Compute the previous state, $\vec{m}[t-1]$.

Solution: We begin by decomposing the state into a linear combination of eigenvectors. Solving a straightforward gaussian elimination yields that $\vec{m}[t]$ may be written as:

$$\vec{m}[t] = 3\vec{v}_1 + \vec{v}_2 + 2\vec{v}_3 \tag{4}$$

Since the defining property of an eigenvector is that it is scaled linearly by its eigenvector, we can instead divide by the eigenvalue to compute the previous version. In other words:

$$\vec{m}[t-1] = \frac{3}{1}\vec{v}_1 + \frac{1}{2}\vec{v}_2 + \frac{2}{0.5}\vec{v}_3 \tag{5}$$

This yields:

$$\vec{m}[t-1] = \begin{pmatrix} 4 \\ 6.5 \\ 12 \end{pmatrix} \tag{6}$$

8. Building Spaces (10 Points)

(a) (4 points) What is the nullspace of the following matrix?

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

Solution: To solve for the nullspace, we must solve for the values of the vector $\vec{v}$ that solve the equation $\mathbf{M}\vec{v} = \vec{0}$. We see that the matrix given to us is already in RREF form, so we don't need to do any further row-reduction, and can convert straight to algebraic form to solve for $\vec{v}$.

Let's represent the entries of $\vec{v}$ as $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Setting $\mathbf{M}\vec{v} = \vec{0}$ then results in the following set of equations:

$$1x + 0y + 2z = 0$$
$$0x + 1y - 3z = 0$$

From the RREF form of the matrix, we see that the first two columns have pivots, but the third column does not, so z is a free variable. We can therefore set $z = t$ for some parameter $t \in \mathbb{R}$. Writing everything in terms of t, we end up with the following system of equations:

$$x + 2t = 0$$
$$y - 3t = 0$$
$$z = t$$

Formatting this into vector form, we get:

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} t$$

Because t can take on an value in $\mathbb{R}$, we can say that $\vec{v} \in \text{span}\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$, so the nullspace of the matrix is $\text{span}\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$.

(b) (6 points) Assume for a system of equations $A\vec{x} = b$, we find that the set of solutions for $\vec{x}$ can be written as

$$X = \left\{ \vec{x} \;\middle|\; \vec{x} = \vec{c} + \alpha \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

You want to pick a value of $\vec{c}$ that will ensure that the set of solutions X forms a vector subspace of $\mathbb{R}^3$. For each of the following possible values for $\vec{c}$, select whether it would make X a subspace of $\mathbb{R}^3$. Justify your answer by showing whether or not it satisfies the 3 properties of a subspace.

i. $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- ● X is a subspace
- ○ X is *not* a subspace

Solution:

Yes. In this case, since $\vec{c} = \vec{0}$, the set of possible values for $\vec{x}$ reduces to $\alpha \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, which is equivalent to stating that $X = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$.

The span of a set of vectors always forms a subspace. However, we can check this explicitly by verifying the properties of a subspace. First, we can check closure under scalar multiplication. Let $\vec{x} = \alpha \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$. If we scale $\vec{x}$ by a real number γ, we can show that closure holds under scalar multiplication:

$$\gamma \vec{x} = \gamma \left(\alpha \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) = \left((\gamma\alpha) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (\gamma\beta) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \in X$$

Similarly, let $\vec{x}_1 \in X$ be $\alpha_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \beta_1 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, and let $\vec{x}_2 \in X$ be $\alpha_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$. Then,

$$\vec{x}_1 + \vec{x}_2 = \alpha_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \beta_1 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = (\alpha_1 + \alpha_2) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (\beta_1 + \beta_2) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \in X$$

Therefore, the system is also closed under vector addition. Finally, we can verify that the set contains the zero vector, because we can select $\alpha = 0$ and $\beta = 0$.

ii. $\vec{c} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

○ X is a subspace

● X is *not* a subspace

Solution:

iii. **No**. The simplest way to see this is that the set X will not contain the zero vector with this value of $\vec{c}$. No matter what values we choose for α and β, the first entry of $\vec{x}$ will always be 2. Therefore, this does not form a subspace.

iv. $\vec{c} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$

- ● X is a subspace
- ○ X is *not* a subspace

Solution: **Yes**. In this case, $\vec{c} \in \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$. Therefore, the set of possible values of $\vec{x}$ reduces to $\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$, which we already proved was a subspace in part (i).

9. Some Proofs (10 points)

(a) (5 points) Prove that if there are two unique solutions $\vec{x}_1$ and $\vec{x}_2$ to the system $A\vec{x} = \vec{b}$, then there are infinitely many solutions $\vec{x}$ to this system.

Solution:

$$A\vec{x}_1 = \vec{b}$$
$$A\vec{x}_2 = \vec{b}$$
$$A(\vec{x}_1 - \vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$$
$$\implies \vec{x}_1 - \vec{x}_2 \in Null(A)$$

Vectors of the form $\vec{x} = \vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2)$ are also solutions to the system $A\vec{x} = \vec{b}$ (for $\alpha \in \mathbb{R}$):

$$A(\vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2)) = A\vec{x}_1 + \alpha A(\vec{x}_1 - \vec{x}_2) = \vec{b} + \vec{0} = \vec{b}$$

Note that $\vec{x}_1 - \vec{x}_2 \neq \vec{0}$ as $\vec{x}_1 \neq \vec{x}_2 \implies \vec{x} = \vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2) \neq \vec{x}_1$ for $\alpha \neq 0$.
Therefore we have constructed an infinite number of solutions.

(b) (5 points) A matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ has eigenvalues λ_1, λ_2, ... λ_n and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Let $\lambda_1 = 0$, $\lambda_2 = 0$ and λ_3, λ_4, ... $\lambda_n \neq 0$. Prove that $\vec{x} \in Null(A)$ for any $\vec{x} \in span\{\vec{v}_1, \vec{v}_2\}$.

Solution:

$$\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 \text{ where } \alpha, \beta \in \mathbb{R}$$

$$\begin{aligned} A\vec{x} &= A(\alpha\vec{v}_1 + \beta\vec{v}_2) \\ &= \alpha A\vec{v}_1 + \beta A\vec{v}_2 \\ &= \alpha\lambda_1\vec{v}_1 + \beta\lambda_2\vec{v}_2 \\ &= \alpha(0)\vec{v}_1 + \beta(0)\vec{v}_2 \\ &= \vec{0} + \vec{0} = \vec{0} \\ &\implies \vec{x} \in Null(A) \end{aligned}$$