EECS 16A Fall 2020 — Some of the Proofs We Have Covered So Far

1. **Note 3 | 3.1.1**

   Prove the following two definitions of Linear Dependence are equivalent:

   Definition 3.1: A set of vectors $\{\vec{v}_1, ..., \vec{v}_n\}$ is linearly dependent if there exist scalars $\alpha_1, ..., \alpha_n$ such that $\alpha_1 \vec{v}_1 + ... + \alpha_n \vec{v}_n = 0$ and not all α_i's are equal to zero.

   Definition 3.2: A set of vectors $\{\vec{v}_1, ..., \vec{v}_n\}$ is linearly dependent if there exist scalars $\alpha_1, ..., \alpha_n$ and an index i such that $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$. In words, a set of vectors is linearly dependent if one of the vectors could be written as a linear combination of the rest of the vectors.

2. **Note 3 | 3.1.3**

   Prove the following theorem:

   Theorem 3.1: If the system of linear equations $\mathbf{A}\vec{x} = \vec{b}$ has an infinite number of solutions, then the columns of A are linearly dependent.

3. **Note 4 | Example 4.1 (Example of Constructive Proof)**

   Prove that span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$

4. **Note 4 | Example 4.2 (Example of Proof By Contradiction)**

   Prove the following theorem by contradiction:

   Theorem 4.1: If the columns of $\mathbf{A}$ in the system of linear equations $\mathbf{A}\vec{x} = \vec{b}$ are linearly dependent, then the system does not have a unique solution.

5. **Note 4 | Example 4.3**

   Let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be a set of linearly dependent vectors in $\mathbb{R}^n$. Take any matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. Prove that the set of vectors $\{\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2, ..., \mathbf{A}\vec{v}_n\}$ is linearly dependent.

6. **Note 4 | Example 4.4 (Example of Direct Proof)**

   Assume that vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_1 + \vec{v}_2$ are all solutions to the system of linear equations $\mathbf{A}\vec{x} = \vec{b}$. Prove that $\vec{b}$ must be the zero vector.

7. **Discussion 3A | Q1**

   Given some set of vectors $\{\vec{v}_1, \vec{v}_2, ...\vec{v}_n\}$, show the following:

   (a) span $\{\vec{v}_1, \vec{v}_2, ...\vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, ...\vec{v}_n\}$, where α is a non-zero scalar. In other words, we can scale our spanning vectors and not change their span.

   (b) span $\{\vec{v}_1, \vec{v}_2, ...\vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, ...\vec{v}_n\}$. In other words, we can replace one vector with the sum of itself and another vector and not change their span.

8. **Discussion 3A | Q2 Part 3**

   The distributivity property of matrix-vector multiplication holds for any vectors and matrices. Show for general $\mathbf{A} \in \mathbb{R}^{2 \times 2}$ and $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ that $\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2$.

9. Note 6 | 6.1.1

Prove the following theorems:

(a) **Theorem 6.1**: If $\mathbf{A}$ is an invertible matrix, then its inverse must be unique.

(b) **Theorem 6.2**: If $\mathbf{QP} = \mathbf{I}$ and $\mathbf{RQ} = \mathbf{I}$, then $\mathbf{P} = \mathbf{R}$. The matrix P can be thought of as the "right" inverse of $\mathbf{Q}$ and the matrix $\mathbf{R}$ can be thought of as the "left" inverse of $\mathbf{Q}$.

10. Note 6 | 6.2

Prove the following theorems:

(a) **Theorem 6.3**: If a matrix $\mathbf{A}$ is invertible, there exists a unique solution to the equation $\mathbf{A}\vec{x} = \vec{b}$ for all possible vectors $\vec{b}$.

(b) **Theorem 6.4**: If a matrix $\mathbf{A}$ is invertible, its columns are linearly independent.

11. Homework 4 | Problem 6(f)

Consider a system consisting of k reservoirs such that the entries of each column in the system's state transition matrix sum to one.

Prove that if s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

12. Discussion 4B | Q3

Is the set $V = \left\{ \vec{v} \middle| \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, where $c, d \in \mathbb{R}$, a subspace of $\mathbb{R}^3$?

13. Note 9 | 9.6.1

Prove the following theorem:

Theorem 9.1: Given two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two different eigenvalues λ_1 and λ_2 of a matrix A, it is always the case that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

14. (Proof Out of Scope) Note 9 | 9.6.2 (Proof By Induction)

Prove the following theorem:

Theorem 9.2: Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be eigenvectors of an $n \times n$ matrix with distinct eigenvalues. It is the case that all the $\vec{v}_i$ are linearly independent from one another.

The proof of this theorem is out of scope, but is presented anyway just for reference for those who are interested.

