EECS16A: A Compendium of Notes and Practice Problems

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April 9, 2022

Contents

Part I

Introductory Content

Introduction

Welcome! I hope you find this book an effective and worthwhile way to help learn the content in this course. This resource contains relevant equations, insights, and information associated with each week's content, solutions to the online practice problem sets, and (at some point will hopefully also contain) other problems that I create.

Each chapter in Online Practice Sets is associated with one of the online practice sets, as found here. These chapters contain two sections: RELEVANT INFORMATION and PROBLEMS.

The RELEVANT INFORMATION sections will be condensed versions of the official course notes (as found on the official course website). These sections serve merely as a reference for what I believe are the most important concepts from each lecture. Notably, these notes often do not contain the motivating context that make the content interesting. I've made it more about the facts, certain useful insights that I believe are critical to aid in visualization and broader understanding, and execution of problem-solving techniques. In that sense, it is largely incomplete. Furthermore, the notes will assume some familiarity with the course content at times, and *are not a substitute for attending lecture and discussion*. Furthermore note that *any concepts introduced in this book but not covered in lecture or in the course notes are not valid for use in an exam.*

The PROBLEMS sections will contain detailed explanations for each problem, with the goal being to enable the reader to solve harder problems dealing with similar concepts. See Conventions for more detail on how to best use these sections.

The goal in writing this text is to make a centralized source for the online practice set solutions that assume little content familiarity, and are detailed enough that a student can actually learn from the solution, rather than merely using it to reinforce prior knowledge. A list of all parts, chapters, and sections are at the Contents for quick navigation.

Conventions

Colors

Magenta: Internal clickable jump-reference to equation, figure, etc. or topic header.

Red: Clickable hyperlink to external website/resource.

Orange: Answer. Just the answer, no frills.

Blue: Quick solution. Streamlined approach that often includes only the minimum necessary while still remaining complete. When there is no long solution, this might include some insights beyond the strict scope of the problem-solving techniques. Occasionally tells the reader to reference the long solution. Meant to be used as a quick check of your work and approach.

Green: Long solution. Includes all steps except possibly when redundant from QUICK SOLUTION or from previous problems. It contains much more insight and intuition, and each step is more thoroughly explained. Occasionally redundant with the QUICK SOLUTION. I recommend that everyone read these, even if you know how to solve the problem. It may include another approach or ideas you haven't considered before but could be useful for exams where similar problems could arise. Often also derives important concepts and points to useful places in the RELEVANT INFORMATION sections.

Font Styles

SMALL CAPS:	Name of topic/header/subsection.
italics:	primarily for chapter/section headings. When used in text, indicates notes and ideas to pay extra attention to.
bold:	Sparingly used in text as a subdivision/sub-header. In equations, almost always used for matrices.
typewriter font:	Indicates matrix operations or properties including null-spaces (null), column- or row-spaces (col or (row)), dimensions (dim), REF, RREF, and range. For rank and determinant this style indicates application of an operation (such as (rank($\mathbf{A}$) or (det($\mathbf{A}$)), and not just a reference to the concept).

Other Conventions/Shorthands

$\underset{r \times c}{\mathbf{A}}$: As a convenient shorthand, when matrices or vectors have an expression below them the form r × c, these are that object's row × column dimensions.

here's some content explaining a concept[1]: That small number is a margin note (see right)! If there's relevant information that doesn't fit into the flow of a solution, or provides additional or out-of-scope context, I put it in a margin note.

[1] this is a margin note for the content.

Part II

Online Practice Sets

Practice Set 0: Systems of Equations and Gaussian Elimination

Relevant Equations/Information

Properties of Scaling:

1. Scaling: Does scaling the input scale the output by the same amount?

$$f(\alpha x) = \alpha f(x) \tag{1}$$

2. Superposition: Does applying a function to the sum of two (or more) input variables have the same effect as if we applied the function to the inputs separately?

$$f(x_1 + x_2) = f(x_1) + f(x_2) \tag{2}$$

3. The above two combined:

$$f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2) \tag{3}$$

Linear Function: A function mapping any number of inputs to an output is written as $f : \mathbb{R}^N \to \mathbb{R}$. Here, f is some input $\to$ output mapping, taking a variable (or multiple) and returning a value. If f is linear, then we can write it in the following form:

$$f(x_1, x_2, \ldots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \tag{4}$$

That is to say, every linear function can be written as a scaled sum, or a weighted average, of the inputs.[2]

Affine Function: An affine function has the following form:

$$f(x_1, x_2, \ldots, x_n) = K + \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \tag{5}$$

Note the difference between affine and linear functions; there is additional constant K here that does not scale with the input linearly. That is, multiplying the input by some scalar will not affect the value of K, so the overall function does not satisfy eq. (1). Only if $K = 0$ would the function be linear. Consider $f(x) = x + 1$; this example demonstrates that even if a function *looks* like a line, it might not be linear.

[2] The scalars α_i are intrinsic to the function, and do not depend on the specific values of the inputs.

Linear Equation: If we have a linear function $f : \mathbb{R}^N \to \mathbb{R}$ and some constant b, then an equation of the following form is linear:[3]

$$f(x_1, x_2, \ldots, x_n) = b \tag{6}$$

A *system* of linear equations is a collection of equations in this form.

Augmented Matrix Representation of a System: Given many linear equations, we can use an *augmented matrix* as an equivalent representation for the system that is easier to mathematically manipulate and solve. Take this system of linear equations:[4]

$$f_1(x_1, x_2, \ldots, x_n) = b_1 \implies \alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n = b_1$$

$$f_2(x_1, x_2, \ldots, x_n) = b_2 \implies \alpha_{21}x_1 + \alpha_{22}x_2 + \ldots + \alpha_{2n}x_n = b_2$$

$$\vdots$$

$$f_n(x_1, x_2, \ldots, x_n) = b_n \implies \alpha_{n1}x_1 + \alpha_{n2}x_2 + \ldots + \alpha_{nn}x_n = b_n$$

Noticing that the variable names columns are redundant and the only differences across equations are the coefficients, we convert the above, matching coefficients to variables (columns), to the following augmented matrix form:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} & b_1 \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} & b_n \end{bmatrix}$$

Gaussian Elimination Procedure: We perform Gaussian Elimination on a system's augmented matrix representation to determine what its solution space looks like. To perform Gaussian Elimination on any given matrix $\underset{m \times n}{\mathbf{A}}$, follow the steps below. There are 2 major parts: the *forward pass* (operations typically work "down" the matrix rows) and the *backward pass* (which work "up" the matrix rows).

Forward Pass: For each row r of the total n starting at the top, perform the following:

1. Swap row r with another row as needed so the leading entry is as far left as possible.

2. Scale row r so that the leading entry[5] is 1

3. For each row below row r, add a multiple of row r so that all entries below the leading entry of row r are equal to zero.

Backward Pass: For each row r of the total n starting at the bottom, add a multiple of row r so that all entries above the leading entry of row r are equal to

[3] Note the relationship between a linear *function* and a linear *equation*. An affine function is linear because we can move the constant term K and group it with b as in eq. (6). In this way, we "separate out" the linear function from the constant term. Another approach is to consider that eq. (6) is identical to eq. (4) only if $b = 0$. These realizations are important for the following questions.

[4] If you haven't seen it before, $\implies$ means 'implies'.

[5] The *leading entry* is the leftmost (nonzero) entry in a given row of a matrix.

zero. After the forward pass, the matrix is in row-echelon form (or REF), and after the backward pass, it is in reduced row-echelon form (or RREF[6]). For many kinds of worked examples, please check the course notes.

[6] Interesting fact: each augmented matrix has a unique RREF.

Gaussian Elimination: Valid Operations: Here's a valid list of *elementary row operations* to perform on rows of a matrix during Gaussian Elimination:

1. Scale an entire row[7] by some nonzero constant

2. Swap two rows.

3. Add a multiple of one row to another.

[7] Remember here that a row represents an equation in the original system.

Some Other Definitions:

- *Pivot* $\equiv$ leading entry, these terms are used interchangeably.

- *Basic Variable*: variables corresponding to columns that contain leading entries in the REF form.

- *Free Variable*: all other columns (not corresponding to basic variables).[8]

- *Consistent System* (of linear equations): has at least 1 solution, infinitely many if we have ≥ 1 free variable, and exactly 1 if there are no free variables.

- *Inconsistent System*: has no solutions. Identified by looking at the RREF form of an augmented matrix corresponding to a system; if there's a row of the form $\begin{bmatrix} 0 & 0 & \ldots & 0 & | & 1 \end{bmatrix}$, the system is inconsistent. This is because such a row represents the equation $0x_1 + 0x_2 + \ldots + 0x_n = 1 \implies 0 = 1$.

[8] For example, in the matrices below, pivots are circled in red, basic variables are in red text, whereas free variables are in blue text.

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \\ \end{array}$$
$$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 \\ 0 & \textcircled{1} & 3 & 4 & 5 \\ 0 & 0 & \textcircled{1} & 4 & 5 \\ 0 & 0 & 0 & \textcircled{1} & 5 \end{bmatrix}$$

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ \end{array}$$
$$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4 & 5 & 6 & 7 & b_1 \\ 0 & 0 & \textcircled{1} & 4 & 5 & 6 & 7 & b_2 \\ 0 & 0 & 0 & \textcircled{1} & 5 & 6 & 7 & b_3 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 7 & b_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} & b_5 \end{bmatrix}$$

Problems

✠ **Problem:** Is the equation $x + 2y = 4z$ linear?

Answer: Yes.

Quick Solution: First, we write the equation in the form $f(x, y, z) = b$. Here, $x + 2y - 4z = 0$, and eq. (6) says that the equation is linear only so long as the *corresponding function* on the left side is linear. To check that, we apply eq. (4); if we pattern match ($x = x_1$, $y = x_2$, and $z = x_3$), then our constants are simply $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = -4$. We then have linearity by definition!

Long Solution: To check linearity of the function on the left side, we can also perform a more rigorous analysis for completeness, so we know how to apply the properties when unfamiliar questions arise.

A function is linear if it satisfies the properties of linearity, as in eq. (1) and eq. (2) (summarized by eq. (3)). Here, the left hand side is:

$$f(x, y, z) = x + 2y - 4z$$

If we rewrite each variable with a scaled and summed version of itself to match the form of eq. (3), do we satisfy the properties? Let's see:

$$(\alpha x_1 + \beta x_2) + 2(\alpha y_1 + \beta y_2) - 4(\alpha z_1 + \beta z_2) \overset{?}{=} \alpha(x_1 + 2y_1 - 4z_1) + \beta(x_2 + 2y_2 - 4z_2)$$

$$\alpha x_1 + 2\alpha y_1 - 4\alpha z_1 + \beta x_2 + 2\beta y_2 - 4\beta z_2 \overset{?}{=} \alpha(x_1 + 2y_1 - 4z_1) + \beta(x_2 - 2y_2 - 4z_2)$$

$$\alpha(x_1 + 2y_1 - 4z_1) + \beta(x_2 + 2y_2 - 4z_2) \overset{\checkmark}{=} \alpha(x_1 + 2y_1 - 4z_1) + \beta(x_2 + 2y_2 - 4z_2)$$

Make sure you understand why the first line is the primary question we ask in determining linearity; the rest is rearranging terms and algebra. Indeed, we arrive at an equality; the equation is linear.

✠ **Problem:** Is the equation $\sin(x) - 2 = 6$ linear?

Answer: No.

Quick Solution: Simply put, applying eq. (4) and putting the given equation in the form $\sin(x) = 8$, we observe what makes this equation nonlinear: the corresponding function on the left is nonlinear since the scaling factor for x is not a constant.

Long Solution: See eqs. (1) to (3) for a summary on the properties of linearity. In the form $f(x_1, x_2, \ldots, x_n) = b$, we have:

$$\sin(x) = 8$$

Does the function on the left satisfy the relevant properties?

$$\sin(\alpha x_1 + \beta x_2) \overset{?}{=} \alpha \sin(x_1) + \beta \sin(x_2)$$

We can already see that the property will not hold for all choices of α, β, x_1, and x_2. Instead of going through and applying the properties rigorously, invoking sum-of-angle formulas and the like, we observe the following:

$$\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

whereas:

$$\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2} \,[\neq 1]$$

There are numerous counterexamples to dispute this equation's linearity. Or, you could simply have argued (accurately) that since $\sin(x)$ does not "look like a line," the function couldn't be linear.[9]

[9] But beware of this sort of intuition in general; when we get to more abstract concepts like vector spaces and such, the properties and definitions are far safer to rely upon.

✠ **Problem:** Is the equation $\sum_{i=1}^{50} i \cdot x - e^{-3}y = \sin\left(\frac{\pi}{3}\right)$ linear?

Answer: Yes (perhaps surprisingly!)

Quick Solution: Once we see that the summation is just a fancy constant, the exponential is another fancy constant, and the right side is also just a constant, the form matches eq. (6) closely. From there, we apply eq. (4) to the left side, and note that since the left is a linear function, the equation is also linear.

Long Solution: This equation may seem intimidating, and its sheer complexity

and notation may lead one to incorrectly say that it isn't linear. But let's break it down using the properties we know.

First, the right-hand side is just a constant (taking the sine of some angle), so that's taken care of. We also have a constant (e^{-3}) scaling y. Now what about this summation?[10] We just add all the numbers from 1 to 50 and whatever that ends up being,[11] we scale x by that value.

Let's invoke eq. (4) on the left side of the equation. The function on the left is linear and we just have a constant on the right, so according to eq. (6), we have a linear equation!

✠ **Problem:** Write the following system of equations in matrix form:

$$\begin{cases} 2x - 3y = 1 \\ 3x + \ y = -2 \end{cases}$$

Answer:

$$\begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Quick Solution: We simply match coefficients to variables and observe the underlying matrix-vector structure. The system looks very similar to the $\mathbf{A}\vec{x} = \vec{b}$ form, and for good reason.

Long Solution: We note that the form of a matrix equation representing a linear system of equations is as follows: $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A}$ is the multiplying matrix that contains all the coefficients, $\vec{x}$ is the vector containing the unknown variables $x_1 \ldots x_n$, and $\vec{b}$ is the output vector containing the values that each row of summed and scaled variables evaluates to.

Each row in the $\mathbf{A}$ matrix corresponds to the coefficients for one equation, and the variables they multiply can be found in the $\vec{x}$ vector. Here, the expectation is to be able to pattern-match. As a quick sanity check, if it isn't evident that the -3 coefficient scales the variable y for the first equation, then it will likely help to wait until we get to the mechanics of matrix multiplication. A simple formula is found in eq. (7), and the more generalized treatment of matrices vectors can be found in the course notes.[12]

✠ **Problem:** Is it *true* or *false* that a system of 3 equations with 2 free variables has solutions along a line?

Answer: False. They lie on a 2D plane.

Quick Solution: Since there are 2 free variables, we effectively have 2 degrees of freedom (in being able to arbitrarily vary those two parameters and still satisfy the system of equations). A line would imply 1 degree of freedom; a plane encompasses 2.

Long Solution: What does it mean for a system of equations to have a free

variable? It means that variable doesn't add any constraints on the solutions to that system. Said another way, no matter what the value of a free variable is, the values of the other (*basic* or *constraining*) variables will determine what the solution space looks like for the system.

In this case, we have 3 equations, and we know 2 of the variables involved are free. The remaining variables are basic, which means that they have specific values set by the system of equations because they determine the set of solutions. We can vary one free variable along an infinite line and vary the other free variable along a different line; these two lines define a plane. Because we can vary *both* free variables however we'd like and still have a solution to the system, the solution space would not be a line, which is a 1D object (only 1 degree of freedom). We actually have the next higher dimension object[13] as our solution space, which is a plane.

Something to note; based on the fact that we have 3 equations and 2 free variables, what is the maximum number of total variables we can have in this system? Each variable is either basic or free. Basic variables correspond to columns containing pivots, and note that each pivot can be thought of as belonging to *both* a row and a column. With 3 equations, we have only 3 rows; this means that we can have no more than 3 pivots (and 3 basic variables). Including our 2 free variables from before, this means the total variable count is capped at 5.

[13] Nuance for the interested: The set of solutions will always be a plane, since 2 of the variables are free. But one question is, *what dimension of space does the plane lie in?* This entirely depends on the number of *total* variables in the system's equations. In a simple case, we may have 3 total variables, so we have a 2D plane in 3D space. If we have 5 variables, though, we have a (much harder to visualize) 2D plane residing in 5D space!

⚔ **Problem:** We have the following row-reduced augmented matrix, obtained by performing Gaussian Elimination. How many solutions does the underlying system have?

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Answer: ∞

Quick Solution: We note that there is a free variable (x_2) in the system, so we have an infinite number of solutions.

Long Solution: Call the variables x_1, x_2, and x_3. The last row indicates that x_3 has a unique, defined value: $2x_3 = 1 \implies x_3 = \frac{1}{2}$. Now, given x_3, we plug its value into the top row ($3x_1 - x_2 + 2x_3 = 1$) and find that $3x_1 - x_2 = 0$, or $3x_1 = x_2$. We have a free variable, and the system has infinitely many solutions (one for each value of the free variable). Note that if we select a value for either x_1 or x_2, the value for the other variable is automatically determined, so we have only 1 free variable, not 2.

Alternate Solution: Note that we can go one step further and convert this matrix to RREF. We then note that columns containing leading entries correspond to basic variables, and the rest contain free variables. So we convert:

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Note that columns 1 and 3 have leading entries, and column 2 doesn't. So we arrive at the same conclusion, that the system has infinite solutions because x_2 is a free variable.

✠ **Problem:** We have the following row-reduced augmented matrix, obtained by performing Gaussian Elimination. How many solutions does the underlying system have? (Note: this is similar to the question with a slightly different matrix.)

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Answer: None

Quick Solution: Note that the last row represents an impossible equation. $0x_3 \neq 1$ no matter what value of x_3 we select! Therefore, we can conclude immediately that the system is inconsistent and has no solutions.

✠ **Problem:** Is it *true* or *false* that a system of equations with more equations than unknowns will always have either infinitely many solutions or no solutions?

Answer: False

Quick Solution: A simple counterexample is as follows:
$$\begin{cases} x = 1 \\ 2x = 2 \end{cases}$$
We have 2 equations, 1 unknown, and only 1 solution.

Long Solution: To approach these kinds of questions, it's best to consider what kind of edge cases to consider, and what the relevant definitions are. Here, one may be tempted to say that because each equation adds information and we have more "pieces of information" than the number of variables we need values for, that the system is *over-determined* and has infinite solutions.

But what if every equation we have looks like $0x_1 = 1$, $0x_2 = 2$, etc? Then we can have no solutions. But the key realization to make is that some equations *may not* add anything new to the system. Consider the counterexample given in the QUICK SOLUTION above; the second equation is exactly the same as the first except for a scaling factor, so we *effectively* only have 1 combined piece of information. This idea of redundancy will be discussed much more thoroughly in the sections on linear dependence and independence.

✠ **Problem:** Perform Gaussian Elimination (explicitly, using an augmented matrix) on the following system of equations to find x, y, and z.
$$\begin{cases} 10x - 6y + 2z = 2 \\ 3x + 2y = 10 \\ -5x + 3y - z = 1 \end{cases}$$

Answer: No solution.

Quick Solution: In performing Gaussian Elimination on the associated matrix, we arrive at an impossible equation. Check the LONG SOLUTION for details!

Long Solution: Here we go! Let's follow the Gaussian Elimination Algorithm steps.

* First, we put the system into augmented matrix form: $\begin{bmatrix} 10 & -6 & 2 & 2 \\ 3 & 2 & 0 & 10 \\ -5 & 3 & -1 & 1 \end{bmatrix}$

* Next, we begin applying row operations.[14]

$$\frac{1}{10}R_1 \to R_1 \implies \begin{bmatrix} 1 & -\frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 3 & 2 & 0 & 10 \\ -5 & 3 & -1 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2-3R_1 \to R_2 \\ R3+5R_1 \to R_3 \end{matrix} \implies \begin{bmatrix} 1 & -\frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{19}{5} & -\frac{3}{5} & \frac{47}{5} \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

* At this point, we notice that the last row is an impossible equation and implies the system was inconsistent to begin with. Therefore, the system has no solutions.

[14] If you'd like to see the entire set of steps until full RREF, or want to experiment with other variations on this problem, please make cautious use of this link. Beware that it has at least one known bug that I've experienced, so its accuracy is not guaranteed.

✠ **Problem:** Solve the following system of equations for x, y, and z:

$$\begin{cases} 2x + y + 3z = 1 \\ x - y + 4z = 2 \\ x + 8y + z = 1 \end{cases}$$

Answer: $x = -\frac{5}{9}$, $y = \frac{1}{9}$, $z = \frac{2}{3}$.

Quick Solution: Standard Gaussian Elimination, with a unique solution at the end. Check the LONG SOLUTION for details!

Long Solution: No requisite solving method is specified here, so one could simply use back-substitution without involving matrices at all. But for practice, we show the method involving Gaussian elimination:

We repeat the same steps that we carried out above:

* Put the system into augmented matrix form: $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 4 & 2 \\ 1 & 8 & 1 & 1 \end{bmatrix}$

* Apply batches of row operations:[15]

$$\frac{1}{2}R_1 \to R_1 \implies \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 1 & -1 & 4 & 2 \\ 1 & 8 & 1 & 1 \end{bmatrix}$$

[15] Alright, fine, I'll stop including these drawn-out matrix operations. Last time, promise!

$$\begin{array}{c} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \implies \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \Big| & \frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{5}{2} & \Big| & \frac{3}{2} \\ 0 & \frac{15}{2} & -\frac{1}{2} & \Big| & \frac{1}{2} \end{bmatrix}$$

$$\begin{array}{c} -\frac{2}{3} R_2 \to R_2 \\ R_3 - \frac{15}{2} R_2 \to R_3 \\ \frac{R_3}{12} \to R_3 \end{array} \implies \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \Big| & \frac{1}{2} \\ 0 & 1 & -\frac{5}{3} & \Big| & -1 \\ 0 & 0 & 1 & \Big| & \frac{2}{3} \end{bmatrix}$$

$$\begin{array}{c} R_2 + \frac{5}{3} R_3 \to R_2 \\ R_1 - \frac{3}{2} R_3 \to R_1 \\ R_1 - \frac{1}{2} R_2 \to R_1 \end{array} \implies \begin{bmatrix} 1 & 0 & 0 & \Big| & -\frac{5}{9} \\ 0 & 1 & 0 & \Big| & \frac{1}{9} \\ 0 & 0 & 1 & \Big| & \frac{2}{3} \end{bmatrix}$$

* Now we read our solutions off the right column directly! $x = -\frac{5}{9}$, $y = \frac{1}{9}$, and $z = \frac{2}{3}$.

✠ **Problem:** Solve the following system of equations for x, y, and z:

$$\begin{cases} 2x - 16y + 4z = -8 \\ x + 12y + 4z = 6 \\ x + 8y - 2z = 4 \end{cases}$$

Answer: $x = 0$, $y = \frac{1}{2}$, $z = 0$.

Quick Solution: If we're really perceptive (and a bit lucky), we notice that the final values depend only on the value of y (the $\vec{b}$ values are each exactly half the coefficient that scales y in each equation). But beyond that, we need to go through and do a more complete analysis.

Long Solution: Instead of doing it out step-by-step, I'll simply include the REF and RREF matrices below, and refer you to the previous 2 problems for the general procedure, which applies here as well. Note that if you prefer to stop at REF (rather than going all the way to RREF[16]) and back-substitute manually to get the final values, that's fine (so long as an exam doesn't specify a method!) Practice will reveal which approach is fastest and most accurate for you.

[16] In some circles, you may hear that REF is obtained with Gaussian Elimination and RREF is arrived at through Gauss-Jordan Elimination. In this course, the naming distinction doesn't matter.

$$\text{REF} = \begin{bmatrix} 1 & -8 & 2 & \Big| & -4 \\ 0 & 1 & \frac{1}{10} & \Big| & \frac{1}{2} \\ 0 & 0 & 1 & \Big| & 0 \end{bmatrix}$$

$$\text{RREF} = \begin{bmatrix} 1 & 0 & 0 & \Big| & 0 \\ 0 & 1 & 0 & \Big| & \frac{1}{2} \\ 0 & 0 & 1 & \Big| & 0 \end{bmatrix}$$

Practice Set 1: Matrix Operations and Linear Dependence

Relevant Equations/Information:

Matrix Multiplication: a_{ij} is the entry in row i, column j. Remember the multiplication of matrices $\left[\underset{m \times n}{\mathbf{A}} \times \underset{p \times q}{\mathbf{B}} = \underset{m \times q}{\mathbf{C}} \right]$ is only defined if $n = p$. Below, we show a simple 2×2 example to illustrate the concept.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{bmatrix} \tag{7}$$

Matrix-Vector Multiplication: Multiplication of a vector by a matrix can be thought of as a *linear transformation* of that vector, very similar to a linear function. Whereas we previously dealt with $f : \mathbb{R}^N \to \mathbb{R}$, taking any number of inputs (which can be represented as the elements of a vector) and returning a single value, we now have the more general $f : \mathbb{R}^N \to \mathbb{R}^M$ which similarly takes any number of inputs but can now return any number of outputs (of course, the dimensions of $\mathbf{A}$ have to be compatible with the transformation.) Linear transformations follow certain properties; recall eqs. (1) to (3).

Matrix-Vector Column-Wise Formulation:[17]

For $\underset{m \times n}{\mathbf{A}} \times \underset{n \times 1}{\vec{x}} = \underset{m \times 1}{\vec{b}}$:

$$b_i = \sum_{j=1}^{n} a_{ij} x_j$$

In this way, we can interpret the $\mathbf{A}\vec{x}$ product in the context of $\mathbf{A}$'s columns:

$$\mathbf{A}\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + A_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + A_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \tag{8}$$

Linear Dependence: Linear Dependence is a concept used to characterize the

[17] For a more complete/generalized treatment of matrices, vectors, and more, including several important properties, see the course notes.

amount of redundancy in a given set of information. It is very closely tied to the idea of free and basic variables as we've already seen, and forms the basis (pun intended) for a large number of important concepts going forward. We present 3 equivalent definitions for Linear Dependence below.[18]

1. A set of vectors $\{\vec{v_1}, \ldots, \vec{v_n}\}$ is linearly dependent if there are corresponding scalars $\alpha_1, \ldots, \alpha_n$ such that:

$$\alpha_1 \vec{v_1} + \cdots + \alpha_n \vec{v_n} = \vec{0} \tag{9}$$

   However, *not all α_i's can be zero*; this combination of all-zero scalars has a special name: the "trivial solution."

2. Vectors $\vec{v_1}, \ldots, \vec{v_n}$ with associated scalars $\alpha_1, \ldots, \alpha_n$ are linearly dependent if, for some index i, we have that $\vec{v_i} = \sum_{j \neq i} \alpha_j \vec{v_j}$. [19] That is, a set of vectors is linearly dependent if one of them can be "composed" by linearly combining some/all of the other vectors.

3. A set of vectors is either linearly dependent or linearly independent. More specifically, consider the sum in eq. (9). If there is a solution to satisfy this equation other than to make all the scalars $\alpha_1 = \cdots = \alpha_n = 0$, (that is, a nontrivial solution) then the vectors are linearly dependent.

Span: Informally, the span of a set S of vectors (which is the only context it really makes sense to talk about span, with edge cases being potentially empty or single-vector sets) is the set of all vectors that can be reached using vectors in S. More formally, it is the set of all linear combinations of vectors in S. How do we get all these combinations? By considering all combinations of α_i! We already have the notion of various α_i scaling the vectors in a set $\{\vec{v_1}, \ldots, \vec{v_n}\}$. We simply formalize this:

$$\text{span}(S) = \text{span}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} \alpha_i \vec{v_i} \mid \alpha_i \in \mathbb{R} \right\} \tag{10}$$

   This can be hard to understand without a visual, so let's take a look at a simple 2D example to clarify. If we have a single vector $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, what's our span? It's all vectors that are scaled versions of $\vec{v_1}$. $\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ is in the span ($\alpha_1 = 0.5$), as is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for ($\alpha_1 = 0$) and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ for ($\alpha_1 = 3$). But $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ are not in the span. Simply put, anything on the infinite line passing through $\vec{v_1}$ is in the span; anything else isn't. fig. 1 to the right clarifies with some more example points.
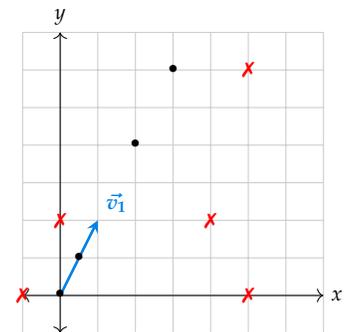
Figure 1: Demonstration of a simple example involving span. Notice the ✗ points are not in the span of $\vec{v_1}$, whereas the • points are.

Problems

✠ **Problem:** Given the following matrices, **A** and **B**, what is the first row of the product $\mathbf{C} = \mathbf{AB}$?

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 10 & 3 & 7 \\ 6 & 4 & 11 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 12 & 3 \\ 1 & 8 & 0 \\ 9 & 1 & 2 \end{bmatrix}$$

Answer: $\begin{bmatrix} 7 & 52 & 3 \end{bmatrix}$

Quick Solution: We apply eq. (7) and find that:

$$c_{1i} = \vec{a_{r1}}\vec{b_{ci}}$$

(dot product of first row of A and i'th column of B). Computing, we find the top row of the resulting matrix is: $\begin{bmatrix} 7 & 52 & 3 \end{bmatrix}$

Long Solution: The first row consists of 3 entries, c_{11}, c_{12}, and c_{13} with c_{ij} being the entry in the i'th row and j'th column. Let's explicitly write out each entry using eq. (7):[20]

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = (1 \times 2) + (5 \times 1) + (0 \times 9) = 7$$
$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = (1 \times 12) + (5 \times 8) + (0 \times 1) = 52$$
$$c_{13} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = (1 \times 3) + (5 \times 0) + (0 \times 2) = 3$$

Combining, we get $\begin{bmatrix} 7 & 52 & 3 \end{bmatrix}$.

[20] Note the dimensions match based on this notation $((1 \times 3) \cdot (3 \times 1))$ and each dot product is valid, yielding a single scalar value.

✠ **Problem:** Is $(\vec{x} + \vec{y})\mathbf{A}$ equivalent to $\mathbf{A}(\vec{x} + \vec{y})$ where $\vec{x}$ and $\vec{y}$ are column vectors in $\mathbb{R}^n$ and $\mathbf{A}$ is an $n \times n$ matrix?

Answer: No, the dimensions only match for $\mathbf{A}(\vec{x} + \vec{y})$.

Quick Solution: One of the expressions isn't a valid multiplication! Since $\vec{x}$ and $\vec{y}$ are column vectors, they have dimensions $n \times 1$. $\mathbf{A}(\vec{x} + \vec{y})$ is valid since $(n \times n) \cdot (n \times 1)$ yields an $n \times 1$ column vector. This matches the $\mathbf{A}\vec{x} = \vec{b}$ form. The other expression is $(n \times 1) \cdot (n \times n)$; note the inner dimensions don't match, so we cannot form the vector-matrix product.

✠ **Problem:** Is it *true* or *false* that if $\mathbf{A}^2 = \mathbf{0}$, the zero matrix, then $\mathbf{A} = \mathbf{0}$?

Answer: Technically, depends on the dimensions of **A**. False *except* in the case of a 1×1 matrix (edge case).

Quick Solution: One approach is to form the counterexample $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and observe that the product is zero but the matrix A itself isn't. But in the case of a 1×1 matrix, if $\mathbf{A}^2 = \mathbf{0}$, then $\mathbf{A} = \mathbf{0}$. So, the answer depends on the dimensions of **A**.[21]

[21] If you solved this question and arrived at *False*, know that this was the intended answer. Disregarding the 1×1 case (often thought of as a scalar rather than a matrix) was an oversight, so the solution on the site is wrong.

Long Solution: This question is difficult to solve rigorously for arbitrary dimensions,[22] but observe the following: Each entry of the resulting matrix product $\mathbf{A}^2$ comes from the dot product of a row and column in the original matrix. A dot product can be zero for many reasons; one of the individual vectors is entirely zero is the most basic case but any combination of values where the sum of multiplications is zero will yield a zero dot product.

[22] The QUICK SOLUTION addresses the edge case of a 1×1 matrix, so we now deal only with 2×2 or larger matrices.

The general realization that the matrix $\mathbf{A}$ doesn't need to be zero in order for $\mathbf{A}^2$ to be zero comes from the fact that a dot product can "produce zeros" from nonzero inputs. This might be counterintuitive since in algebraic equations, if one says $xyz = 0$, one of x, y, or z must be zero, but the logic doesn't carry over. Consider:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

The row-column dot product is zero, but the individual rows indeed aren't zero. Also consider this example:

$$\mathbf{C} = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix} \qquad \mathbf{C}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Try it yourself![23]

[23] For more on this kind of logic, check out Nilpotent Matrices. Note that anything on this page that is out-of-scope or not covered in class notes may not be valid for use in an exam.

✠ **Problem:** If $\mathbf{A}_{4\times 4} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, what is the second column of $\mathbf{A}$?

Answer: $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

Quick Solution: We isolate the second column of A with the single nonzero entry in the vector on the left (if it isn't clear why/how, check out the LONG SOLUTION. So the result is simply the vector on the right.

Long Solution: This question gets at the heart of what it means conceptually to do a matrix-vector multiplication. In an equation of the form $\mathbf{A}\vec{x} = \vec{b}$ each b_i is the inner product of $\mathbf{A}_{ri}$ (i'th row of A) and $\vec{x}$. But notice that the multiplying vector here has a "sifting" property; it nullifies any entries in $\mathbf{A}$ that don't match up to the second entry.

Which entries of $\mathbf{A}$ always end up getting multiplied by 1? The ones in the second column.[24] Ultimately, if these are the only entries that get preserved in the sifting, then $\vec{b}$ entirely consists of entries from $\mathbf{A}_{c2}$. So we directly know what the

[24] For great visualizations on this "mapping," check out the course notes.

second column of A is.

✠ **Problem:** The effects of a fighter jet performing a Barrel Roll, an Immelman, or a Nose Dive can be represented by matrices **A**, **B**, and **C**, respectively. The jet's current state is represented by $\vec{x}$. What expression describes the jet's state after the pilot does a Nose-Dive, Barrel Roll, and Immelman, in that order?

Answer: $\mathbf{BAC}\vec{x}$

Quick Solution: We compose the given operations (**C**, **A**, **B**) by left-multiplying the state vector by the matrices in order, and we arrive at $\mathbf{BAC}\vec{x}$.

Long Solution: Matrix-vector operations that represent some transformation of a state are carried out as a left-multiplication of the state vector (in this case $\vec{x}$) by the matrix to get the state after the operation. Here, we first perform a Nose-Dive, so we left-multiply $\vec{x}$ by **C** to get $\mathbf{C}\vec{x}$. We then perform a Barrel Roll, so we left-multiply $\mathbf{C}\vec{x}$ by **A** to get $\mathbf{AC}\vec{x}$. Finally, we left-multiply $\mathbf{AC}\vec{x}$ by **B** (Immelman) to get the final state vector: $\mathbf{BAC}\vec{x}$.

✠ **Problem:** Are the vectors $\vec{a}$ and $\vec{b}$ below linearly independent?

$$\vec{a} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Answer: Yes.

Quick Solution: There are only two vectors, and neither is a scaled version of the other. So they are linearly independent.

Long Solution: For this small example, even the LONG SOLUTION isn't that long. But to be more rigorous, we can check if we satisfy eq. (9); that is, can any scalars α_1, α_2 can be found such that $\alpha_1 \vec{a} + \alpha_2 \vec{b} = 0$? We formulate this question as:

$$\alpha_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$$

$$\begin{cases} 4\alpha_1 - 1\alpha_2 = 0 \\ 3\alpha_1 + 2\alpha_2 = 0 \end{cases}$$

Solving using standard techniques (substitution, Gaussian elimination, whatever you prefer), we find that $\alpha_1 = \alpha_2 = 0$. In other words, by our 3rd definition of linear dependence, we only have the trivial solution (no nontrivial solution), so the vectors are linearly independent.

✠ **Problem:** Are the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ below linearly independent?

$$\vec{a} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Answer: No.

Quick Solution: We can find that $3\vec{a} - \vec{b} + 5\vec{c} = 0$, and since this is a nontrivial solution to eq. (9), the vectors are linearly dependent.

Long Solution: As before, we set up the corresponding system of equations:

$$\alpha_1 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\begin{cases} 2\alpha_1 + \alpha_2 - \alpha_3 = 0 \\ 2\alpha_1 + 6\alpha_2 = 0 \\ -\alpha_1 + 2\alpha_2 + \alpha_3 = 0 \end{cases}$$

And we solve! The solution to this system is $\alpha_1 = 3, \alpha_2 = -1, \alpha_3 = 5$. And since this is a nontrivial solution, the original vectors are linearly dependent.

✠ **Problem:** For some matrix $\mathbf{A}$, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$, where $\vec{x}_1 \neq \vec{x}_2$. Are the columns of A are linearly independent, or does it depend on the specific matrix?

Answer: No, the columns are linearly dependent.

Quick Solution: We notice that $\mathbf{A}\vec{x}_1 - \mathbf{A}\vec{x}_2 = \mathbf{A}(\vec{x}_1 - \vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$. That is, $\vec{0}$ is a linear combination of the columns of $\mathbf{A}$. So these columns, by eq. (9), are dependent.

Long Solution: This question tests your conceptual understanding of matrix-vector multiplication, with the added nuance of considering linear dependence. Previously, we've considered how each entry in the result, $\vec{b}$, arises from multiplying a row in $\mathbf{A}$ by $\vec{x}_1$ or $\vec{x}_2$. But there's another perspective on the same idea which allows us to view the composition of the $\mathbf{A}\vec{x}_i$ product in terms of the columns of $\mathbf{A}$.

We saw a hint of this in the two previous questions; notice how we had scalars multiplying column vectors, which we then added together and set equal to $\vec{0}$. When there *is a nontrivial solution*, these columns were found to be linearly dependent, and if there was *only the trivial solution*, the columns were linearly independent.

Now, we have that $\mathbf{A}\vec{x}_1 - \mathbf{A}\vec{x}_2 = \mathbf{A}(\vec{x}_1 - \vec{x}_2) = \vec{0}$, but let's take a closer look at why exactly this says anything about the columns of $\mathbf{A}$. We can give the difference between $\vec{x}_1$ and $\vec{x}_2$ a name $\vec{\delta}_x = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}$. Because $\vec{x}_1 \neq \vec{x}_2$, there must be some

nonzero entries in $\vec{\delta}_x$ (not all δ_i's $= 0$).

Let the columns of $\mathbf{A}$ be $\vec{a_{c1}}, \dots, \vec{a_{cn}}$. Then, notice that according to eq. (8), we can say that $\mathbf{A}\vec{\delta} = \sum_{i=1}^{n} \delta_i \vec{a_{ci}} = \vec{0}$. If this step doesn't make sense, please review the expansion of the $\mathbf{A}\vec{\delta}$ (or $\mathbf{A}\vec{x}$) product.

Then, we can conclude by eq. (9) (as we often seem to do) that the columns of $\mathbf{A}$ are linearly dependent.

✠ **Problem:** Is $\vec{v}$ below in the span of the vectors listed?

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{span} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \right)$$

Answer: No.

Quick Solution: Notice that we cannot express $\vec{v}$ as a linear combination of the vectors in the given set, which, by definition in eq. (10), means $\vec{v}$ is not in the span. But this example is not so trivial that we can visually see it immediately; to show this, we carry out Gaussian elimination of the matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & \bigm| & 1 \\ 0 & 2 & 4 & \bigm| & -1 \\ 1 & 1 & -1 & \bigm| & 1 \end{bmatrix} \tag{11}$$

and show there is no solution.

Long Solution: Why does the approach above work? To learn to perform Gaussian Elimination as a procedure to solve equations like $\mathbf{A}\vec{x} = \vec{b}$ takes hours; to visualize and understand the underlying concepts can take months or more.

We must start with the fact that the vectors in the span can be thought of as the columns of a matrix $\mathbf{A} = \begin{bmatrix} | & | & | \\ \vec{a_{c1}} & \vec{a_{c2}} & \vec{a_{c3}} \\ | & | & | \end{bmatrix}$. Remember eq. (8)? Well, that's exactly the interpretation we need! Each of these vectors is a column in the matrix, which as a whole represents a *transformation in space*. We saw a hint of this in the RELEVANT INFORMATION section, but now go into more detail.

When we say that $\mathbf{A}\vec{x} = \vec{x}'$, it can be seen as the vectors effectively taking a point (x, y, z) and mapping it to some other point (x', y', z'). And we can do this for any of the infinite number of points in 3D space. These *coordinates* are now our scalars! The location of a point determines where it gets mapped to by the transformation.[25]

Overall, then, our goal is to figure out what coordinates (x, y, z) are mapped to $\vec{v}$ (the "head" or arrow-part of which is at point $(1, -1, 1)$)[26] by $\mathbf{A}$. We can have no such points (no solution), a unique point (unique solution), or infinitely many points (infinitely many solutions). Now, we already know how to solve an equation of the form $\mathbf{A}\vec{x} = \vec{x}'$; we know $\mathbf{A}$ and $\vec{x}'$, so we can figure out what $\vec{x}$ is. Hopefully it's clear now that when we solve the augmented matrix in eq. (11),

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[25] Note that if we had more or less vectors of greater or smaller length, this transformation would be "cross-dimensional" (mapping a 3D point to a 5D point or something similar) but we omit this consideration for now.

[26] The point at the head of a vector and the vector itself practically synonymous in most cases, since the tail almost always resides at the origin.

we are trying to find this starting point. When we come to the conclusion of no solution (try this yourself to confirm!), it means there is no such starting point! That is, no matter where in 3D space we try to start in order to map to $\vec{v}$ using this transformation, we fail. So the transformation represented by these vectors is incapable of taking us to $\vec{v}$; $\vec{v}$ is not in the span of the given vectors.

✠ **Problem:** Given $\vec{v_1}$ and $\vec{v_2}$, select all of the following options that are equivalent to span($\vec{v_1}$, $\vec{v_2}$):

1. span($\vec{v_1}$, $\vec{v_1} + \vec{v_2}$)
2. span($\vec{v_1}$)
3. span($\vec{v_1}$, $\vec{v_1} - \alpha\vec{v_2}$)

Answer: 1. and 3.

Quick Solution: If we only have $\vec{v_1}$, we lose "additional reach" given by $\vec{v_2}$. So option 2 is not a good substitute. But the others will still work since we can still reach any of the vectors in our original span; we may simply have to pick different scalars.

Long Solution: When it comes to spans, we can select whatever linear combination of the given vectors we want (any choice of scalars α_1 and α_2). So, in the original span, say we select α_1 scaling $\vec{v_1}$ α_2 scaling $\vec{v_2}$. For the first option, an equivalent choice of scalars is $\alpha_1' = \alpha_2 - \alpha_1$ scaling $\vec{v_1}' = \vec{v_1}$ and $\alpha_2' = \alpha_2$ scaling $\vec{v_2}' = \vec{v_1} + \vec{v_2}$. That is, the spans are the same. A very similar argument holds for option 3, except for slightly different values for each α.

However, in option 2, we actually *lose* information by only having $\vec{v_1}$ in our span as opposed to both $\vec{v_1}$ and $\vec{v_2}$. Consider a simple example where we say each vector resides in 2D space[27]: $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The span of just $\vec{v_1}$ is just the x-axis, but the span of both vectors is all of 2D space. See fig. 2 for a visual aid.

[27] Note the problem never said anything about the dimension of the vectors themselves; we assume two-dimensionality for visual simplicity.
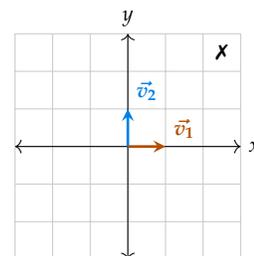
Figure 2: Notice that $\vec{v_1}$ does not encompass the entire span of ($\vec{v_1}$, $\vec{v_2}$). For example, without $\vec{v_2}$, how would one get to ✗ at (2.5, 2.5)?

Practice Set 2: State Transition Matrices and Inverses

Relevant Equations/Information

Linear Transformation: In the previous PRACTICE SET, we discussed the idea of a matrix $\underset{m \times n}{\mathbf{A}}$ as a linear transformation. Effectively, in the equation $\mathbf{A}\vec{x} = \vec{b}$, the matrix itself can be considered a transformation $f : \mathbb{R}^N \to \mathbb{R}^M$ which takes a vector $\underset{n \times 1}{\vec{x}}$ of inputs and returns a vector $\underset{m \times 1}{\vec{b}}$ of outputs (convince yourself the dimensions listed make sense).

In this way, we can think about matrices as state transformations; if we have a list of inputs representing some current state at some timestep n (given by $\vec{x}[n]$), then when a matrix $\mathbf{A}$ operates on that state, it *transforms it* into a new state at the next time step ($\vec{x}[n+1]$). Consider a timestep to be a very small unit of time. Our systems here will be discrete, meaning that the transition of water happens exactly at each timestep, and not between timesteps.[28]

It's best to consider this idea with a classic example; water pumps. Take a look at the basic example in fig. 3 and the more involved example in fig. 4. At each time step, some portion of the water in each pump goes to itself, and some portion goes to each of the other pumps. The general state transition matrix formula for an n-state system (assuming the initial and final state vectors have the same length n) is as follows:

$$\begin{bmatrix} P_{1 \to 1} & P_{2 \to 1} & \cdots & P_{n \to 1} \\ P_{1 \to 2} & P_{2 \to 2} & \cdots & P_{n \to 2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{1 \to n} & P_{2 \to n} & \cdots & P_{n \to n} \end{bmatrix} \tag{12}$$

Conservation: Notice that in the examples in figs. 3 to 4, all of the water goes *somewhere* and none comes up out of thin air; that is, the water is a *conserved quantity*. We don't have any leakage or generation of water in the system. Note that this isn't

Figure 3: The movement of water in this 2-node system of pumps can be represented by this matrix, following the general formula in eq. (12):

$$\begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}$$

[28] But in reality, water is flowing continuously! To model this rigorously, we need linear differential equations, but for now, if the timestep we take is very small, the discrete model is quite good as an approximation.
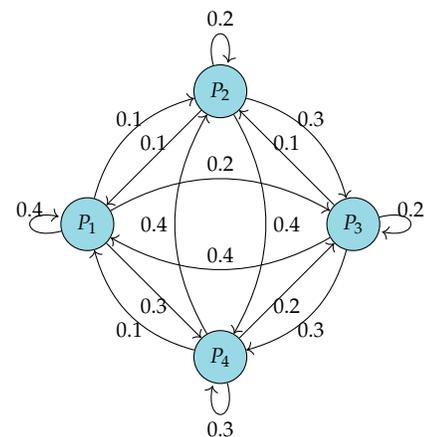
Figure 4: A more involved 4-node state transition example. Yup, it gets messy! Following the general formula in eq. (12):

$$\begin{bmatrix} 0.4 & 0.1 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.3 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 & 0.3 \end{bmatrix}$$

always the case (one could create a state transition matrix describing a single US state and its towns, but people can then travel in and out of the system). The idea of conservation will largely hold true, especially for systems based in physical reality.

How can we easily look at a state transition matrix and check if the corresponding system deals with a conserved quantity? Well, from eq. (12), we see that each *column's* values describe the movement of water *from* a specific node to other nodes. If any column's values do not sum to exactly 1, then something is being lost or created in the system as a whole. In addition, if a specific column's sum is greater than 1, matter is entering the system through that node; conversely, if a specific column sum is less than 1, matter is leaving the system through that node.[29]

Diagram $\rightarrow$ *Matrix*: Given a state transition diagram, we can create the corresponding state transition matrix by reading the values at each arrow, noting the directionality (these are *directed* edges) and populating the rows one by one. Similarly, given a matrix, we can draw the appropriate number of nodes and label arrows going to/from each node with the values as indicated by the matrix.

Conceptual Meaning of a Matrix Inverse: If a matrix $\mathbf{A}$ as a linear transformation tells us how to get from $\vec{x}[n]$ to $\vec{x}[n+1]$, then $\mathbf{A}^{-1}$ tells us how to get from $\vec{x}[n]$ to $\vec{x}[n-1]$.

But we must be careful! The inverse of a matrix is only defined for *square matrices*, and even then, it doesn't always exist. When the inverse exists in the context of state diagrams, it means that if we know what the state is a some time step, then we know what it looks like at the previous timestep. By induction[30], we can figure out what the state looks like at all previous time steps!

Mathematical Definition of a Matrix Inverse: The inverse of a matrix $\mathbf{A}$ (call it $\mathbf{B}$), if it exists, is the matrix such that $\mathbf{AB} = \mathbf{I}$ (in which case $\mathbf{BA} = \mathbf{I}$). Knowing how to express the definition in both ways can come in handy in proofs.

Notice we said *the* inverse, rather than *an* inverse; if the inverse exists, it is assuredly unique.[31].

Inverse for a 2×2 *matrix*:[32]

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{13}$$

[29] Given information about only a single node's column sum, we can never definitely say if the overall system is conservative or not; we only know if it *might* be conservative, based on other nodes.

[30] If you don't know what induction is, don't worry too much, it's a CS70 topic. It basically means that if multiplying our state by an inverse gives us the state 1 timestep before, then we can keep repeating this process and multiplying our state by the inverse n times gives us the state n timesteps before.

[31] See the course notes for a proof of this statement.

[32] I highly recommend you go through the process described below and reduce eq. (14) to eq. (15) to see how this result is derived.

Inverse for an $n \times n$ matrix (using Gaussian Elimination): Start with the augmented matrix of the form in eq. (14) and perform Gaussian Elimination on the left side (carrying the operations along on the right side) until arriving at eq. (15).[33]

$$
\left[\begin{array}{c|c} A & I_n \end{array}\right] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & \ddots & & \vdots & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n1} & \dots & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array}\right] \tag{14}
$$

$$
\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & a_{11}^{-1} & a_{12}^{-1} & \dots & a_{1n}^{-1} \\ 0 & 1 & \dots & 0 & a_{21}^{-1} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{n1}^{-1} & \dots & \dots & a_{nn}^{-1} \end{array}\right] \tag{15}
$$

Intuition Behind Inverses (Using Vectors): In cases where several distinct input vectors can be mapped by the transformation matrix to the same output vector, an inverse cannot exist. Consider why: we have already said that if we *do* have an inverse, it is unique, and we can use it to find *the* input given some output. But here, we are explicitly saying that the output vector could have resulted from more than 1 input vector! That is, we do not know which "backwards branch" to follow. Therefore, we summarize: if A can map two distinct input vectors $\vec{x}_1$ and $\vec{x}_2$ to the same output vector $\vec{b}$, A^{-1} cannot exist. Observe the example in fig. 5.

Inverse: Connection to Linear Dependence: If A is *invertible* (that is, A^{-1} exists), then the columns of A must be linearly independent. Conversely, if a square matrix A has linearly dependent columns, it is not invertible. This can be seen in the matrix A in the caption of fig. 5, for example.[34]

[33] Note that a_{11}^{-1} is *not* the reciprocal of a_{11}; rather, it is the first row, first column element of A^{-1}.

Figure 5: Both vectors $\vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, when transformed by A below, yield the same output vector $\vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, so A^{-1} cannot exist.

$$
A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}
$$

Try it! You should find that the left side cannot be reduced to the identity matrix, and we reach a row of zeros, so the inverse cannot be found. Or using formula eq. (13), we end up dividing by zero on the fraction outside the matrix, also indicating the inverse doesn't exist.

[34] See course notes for a proof!

Problems

✠ **Problem:** Let the state transition matrix A below represent people moving between 3 cities. If the starting state is $\vec{x}[0]$, find $\vec{x}[1]$.

$$
A = \begin{bmatrix} 0.5 & 0.3 & 0 \\ 0 & 0.5 & 1 \\ 0.4 & 0.2 & 0 \end{bmatrix} \qquad \vec{x}[0] = \begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix}
$$

Answer: $\vec{x}[1] = \begin{bmatrix} 110 \\ 200 \\ 80 \end{bmatrix}$

Quick Solution: By definition, **A** describes how the system changes from one step to the next. We're given info about the (input) state at $t = 0$, and want info about the (output) state at $t = 1$. To compute the output after the transformation, we multiply $\vec{x}[0]$ by **A** on the left:

$$\underbrace{\begin{bmatrix} 0.5 & 0.3 & 0 \\ 0 & 0.5 & 1 \\ 0.4 & 0.2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix}}_{\vec{x}[0]} = \underbrace{\begin{bmatrix} 110 \\ 200 \\ 80 \end{bmatrix}}_{\vec{x}[1]}$$

Long Solution: The QUICK SOLUTION is the natural approach here, but alternatively, for a small example, it is also possible to appeal to the physical meaning of eq. (12): City 1 gets 50% of itself, 30% from City 2, and nobody from City 3 [$= 50 + 60 + 0 = 110$]. City 2 gets nobody from City 1, 50% of itself, and all of City 3 [$= 0 + 100 + 100 = 200$]. City 3 gets 40% of City 1, 20% of City 2, and nobody from City 3 [$= 40 + 40 + 0 = 80$].[35]

[35] Note that this is the *exact* same thing that the matrix-vector product is computing for us, but thinking about the physical meaning can help if, for example, we only need a single specific value in the output vector as opposed to the entire vector (aka two problems down).

✠ **Problem:** Considering the same matrix **A** as above, do people stay within the 3 cities?

Answer: No.

Quick Solution: Appealing to the definition and checking the column sums, people are disappearing from City 1 so people are not staying in the system.

Long Solution: Phrased another way, this question is asking if people are *conserved* within the system. In this case, we can appeal to the definition and check the sum of the columns; recall that the values in a specific column i represent the movement of people from City i to all cities in the system. If these proportions do not sum to 1 for *any* i, then we conclude that City i is generating or hemorrhaging people (and these people are not merely coming from or going to other cities in the system, since they are newly introduced to the *system as a whole*).[36] Here, the column sums are as follows: $c_1 : 0.9, c_2 : 1, c_3 : 1$. We notice that $c_1 \neq 1$ so people do not stay within these 3 cities.

[36] For the mathematically inclined, if $\exists j \in \{1, \ldots, n\}$ where $\left(\sum_{i=1}^{n} a_{ij} \right) \neq 1$, the quantity is not conserved in the system.

✠ **Problem:** Given **A** (representing the transfer of water between buckets) and $\vec{x}[n]$ below representing the gallons in each bucket at $t = n$, how much water is in bucket c at $t = n+1$?

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0.1 & 0.4 & 0.5 \\ 0.6 & 0.15 & 0 & 0.2 \\ 0.3 & 0.5 & 0.3 & 0.2 \\ 0 & 0.25 & 0.3 & 0.1 \end{bmatrix} \qquad \vec{x}[n] = \begin{bmatrix} 3 \\ 4 \\ 19 \\ 1 \end{bmatrix}$$

Answer: 8.8 gallons

Quick Solution: Option 1: Compute $\mathbf{A}\vec{x}[n] = \vec{x}[n+1] = \begin{bmatrix} 8.8 \\ 2.6 \\ 8.8 \\ 6.8 \end{bmatrix}$ and select the

amount in bucket c: 8.8 gallons. Option 2: See the LONG SOLUTION!

Long Solution: Note that we only care about the movement of water *from* all buckets *into* bucket 3. We could compute the entire matrix-vector product at once (as in the QUICK SOLUTION and cherry-pick the value we want from the output vector (your computer will make quick work of this computation). But for intuition's sake (and for exams!) we ought to work smart, not hard. Following the suggestion in the margin from 2 problems above, we can work only with the values we care about, found in row 3. Simply compute $3 \cdot 0.3 + 4 \cdot 0.5 + 19 \cdot 0.3 + 1 \cdot 0.2 = 0.9 + 2 + 5.7 + 0.2 = 8.8$ gallons.

⚑ **Problem:** Given the following State Transition Diagram representing search engine traffic by people on the Internet, and corresponding state vector $\vec{x}[t]$, write the State Transition Matrix.
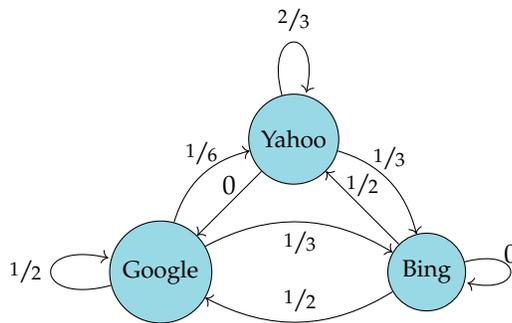
$$\vec{x}[t] = \begin{bmatrix} x_{\text{google}}[t] \\ x_{\text{yahoo}}[t] \\ x_{\text{bing}}[t] \end{bmatrix}$$

Answer: $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$

Quick Solution: Following the pattern in eq. (12), we know there are 3 nodes (ordering given by $\vec{x}$, so we form an empty 3×3 matrix. Then, systematically label each position with the corresponding value in the diagram.

Long Solution: There's not much to add to the QUICK SOLUTION, but here's the explicit matrix labeled for this problem:

$$\begin{bmatrix} P_{g \to g} & P_{y \to g} & P_{b \to g} \\ P_{g \to y} & P_{y \to y} & P_{b \to y} \\ P_{g \to b} & P_{y \to b} & P_{b \to b} \end{bmatrix}$$

✠ **Problem:** Is the internet traffic system described by the diagram above conservative, meaning no people enter or leave the system?

Answer: Yes.

Quick Solution: We apply the definition and check the sum of proportions in each column, as noted at in the RELEVANT INFORMATION section.[37] Since the column sums are all ones, the system is conservative.

[37] See also 3 problems above for a more detailed solution to an almost identical problem.

✠ **Problem:** If the sum of the values in a column for an arbitrary state transition matrix is greater than 1, what does this imply about the corresponding node and system?

Answer: Matter is entering the system through that node over time, the system may or may not be conservative; depends on the other nodes' behaviors.

Quick Solution: We appeal to the definition as presented earlier, and the answer follows directly.

Long Solution: By definition, we note that a column sum of exactly 1 implies conservation of matter *in the system*; nothing is entering or leaving the system through that node. This is a nuance to treat carefully; *a column sum of 1 does not mean the amount of matter at that node stays constant over time.* In fact, this will generally not be the case. It only means that the *system* is not gaining matter over time *through that node.*

Information about the column sum of one node also tells us nothing about other nodes. The system may be conservative, if all other nodes also have column sums of exactly 1. If some columns have sums over 1 and others have sums less than one, who can say what's happening? A look at the specific matrix will be required to dig deeper and make any conclusions.

In this case, a singular column sum greater than 1 indicates that the net contribution of that node with regards to the system is to add matter over time.

✠ **Problem:** Find the inverse of **A** below.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Answer: $\mathbf{A}^{-1} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

Quick Solution: We first form the augmented matrix as described in eq. (14) and when we reduce to eq. (15), we arrive at $\mathbf{A}^{-1}$ as above.

Long Solution: We show the steps here as a complete example of the process of finding an inverse:[38]

[38] However long these take you by hand, take some small comfort in knowing they take me longer to put in LaTeX :-)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 + R_1 \to R_2 \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 + R_2 \to R_3 \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} -R_3 + R_2 \to R_2 \\ -2R_3 + R_1 \to R_1 \end{array} \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

✠ **Problem:** Find the inverse of **A** below. Hint: Reuse some work from above!

$$\left[\begin{array}{ccc} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 0 & -1 & -1 \end{array}\right]$$

Answer: No inverse exists.

Quick Solution: We follow the steps in the previous problem, but we arrive at a row of zeros after the step where we add R_2 to R_3! So the inverse does not exist, since we can't get the identity matrix on the left side.

✠ **Problem:** Suppose $\mathbf{A} = \underset{4\times 2}{\mathbf{B}} \times \underset{2\times 4}{\mathbf{C}}$. Does $\mathbf{A}^{-1}$ exist?

Answer: No.

Quick Solution: We note that based on the dimensions of **C**, it must have more columns than rows, and so its columns cannot all be linearly independent $\implies$ **C** is not invertible $\implies$ $\mathbf{C}\vec{x} = \vec{0}$ must have a solution $\vec{x}$ that is not trivial. By associativity of matrix multiplication,[39] for this same nontrivial $\vec{x}$, $\mathbf{B}(\mathbf{C}\vec{x}) = \vec{0} = (\mathbf{BC})\vec{x} \implies \mathbf{A}\vec{x} = \vec{0}$ and since $\vec{x}$ is nontrivial, **A** cannot be invertible.[40]

[39] See course notes if you need a refresher on this property.

[40] It was so hard not to mention the word null-space, but this will be covered later.

Long Solution: The QUICK SOLUTION is complete in its logic, but skips some steps and justifications, which we elaborate on here.

Let's consider why having more columns than rows implies linear dependence of the columns.[41] The length of a single column vector is given by the number of rows in the matrix (call this r). If we start off with a single column vector of some length, it is a one-dimensional object, describing a 1D line in r-dimensional space. If we try to make the column vectors all linearly independent, then every time we add another column vector, it must be in a different "direction" than all the previous ones. For the second column vector we add, we have $r - 1$ choices for this direction. For the third, we only have $r - 2$, and so on. At some point, we will run out of unique directions to "unlock" with each new vector; the most we

[41] Get ready for some word-based intuition; a more rigorous explanation of all this can be found in the notes, which I recommend reading if this explanation starts to feel unintuitive at any point.

can unlock is r directions representing r dimensions, but once those are used up, we're done!

The takeaway is this; if we have more columns c than rows r, then the first r columns *can* be linearly independent, but the remaining $c - r$ of those necessarily can be created as a linear combination of the r vectors we already have. Why? Because the first r vectors unlocked all of the r-dimensional space the vectors live in. There are no new directions left! By eq. (9) and definition 2 right below it, the full set of c vectors are linearly dependent.

The step where we connected linear dependence of columns to a non-invertible matrix follows directly from the end of the RELEVANT INFORMATION section. The next step uses the 3rd definition of linearity following eq. (9). In the final step, we conclude that nontrivial $\vec{x}$ implies a noninvertible $\mathbf{A}$. This follows from the matrix-vector column-based interpretation in eq. (8), because an invertible $\mathbf{A}$ requires that when $\mathbf{A}\vec{x} = 0$, $\vec{x} = 0$).

✠ **Problem:** Given some $\mathbf{A}$ as the state transition matrix for some system, given $\vec{x}[n]$ at $t = n$, can we always find $\vec{x}[n+1]$ at $t = n + 1$? How?

Answer: Yes. $\mathbf{A}\vec{x}[n] = \vec{x}[n+1]$

Quick Solution: We can always "progress forward" in time given a system's state and corresponding transition matrix. We simply apply the transformation to the desired state to get the transformed state.

✠ **Problem:** Similarly given $\mathbf{A}$ for some system and $\vec{x}[n]$ at $t = n$, can we always find $\vec{x}[n-1]$ at $t = n - 1$? How?

Answer: Not always. Only if $\mathbf{A}^{-1}$ exists, then $\mathbf{A}^{-1}\vec{x}[n] = \vec{x}[n-1]$

Quick Solution: We *cannot* always "track backwards" in time given a system's state and corresponding transition matrix. This is related to the idea of unique input $\rightarrow$ output mappings. If these mappings are unique, then the inverse exists, and we can progress backward. But if not, we cannot go backward since there will be multiple inputs that could have led to an observed output.[42]

[42] See fig. 5 and the corresponding part in the RELEVANT INFORMATION section if you'd like a refresher on this notion of uniqueness.

✠ **Problem:** Suppose we're given state transition matrix $\mathbf{A}$; given $\vec{x}[n]$, can we find $\vec{x}[n-1]$?

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix}$$

Answer: No.

Quick Solution: This problem reduces to finding out if $\mathbf{A}$ is invertible. There's a row of zeros, so it isn't, and so we cannot apply the inverse to go back a timestep from the current state. Also note that the columns are linearly dependent ($c_1 = 2c_2$), so the matrix cannot be invertible.[43]

[43] There are many ways to arrive at the same result, given the connection between concepts like linear dependence, matrix invertibility, matrix columns, and more. Choose whichever approach you prefer!

✠ **Problem:** Given a diagonal matrix **A** where all entries on the diagonal are nonzero but all other entries are zero, is $\mathbf{A}^{-1}$ also diagonal?

Answer: Yes.

Quick Solution: We know the diagonal matrix looks like the left part, and we augment it with I_n to prepare to take the inverse:

$$\left[\begin{array}{cccc|cccc} a_{11} & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & a_{22} & & \vdots & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & \ldots & a_{nn} & 0 & 0 & \ldots & 1 \end{array}\right]$$

Reducing the left to an identity matrix isn't hard; we just divide each row i by the value of a_{ii}! In doing so, the identity matrix entries also become scaled to $\frac{1}{a_{ii}}$, but no new non-diagonal entries are introduced. So the inverse is also diagonal.

✠ **Problem:** Given $\mathbf{A}^N = \mathbf{0}$ (the zero matrix), is **A** invertible?

Answer: No.

Quick Solution: Proceed with Proof By Contradiction:[44] We assume $\mathbf{A}^{-1}$ exists. Then $\mathbf{A}^{N-1} = \mathbf{A}^{-1}\mathbf{A}^N = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$. Using inductive reasoning, $\mathbf{A} = \mathbf{0}$. We now contradict the assumption that **A** is invertible (**0** is not invertible). So our assumption is wrong, **A** is not invertible.

[44] See course notes, this is a common proof technique!

Long Solution: We approach this question by considering what it would mean for $\mathbf{A}^{-1}$ to exist. If it existed, then we could multiply the left side of both expressions[45] by $\mathbf{A}^{-1}$, yielding $\mathbf{A}^{N-1} = \mathbf{A}^{-1}\mathbf{A}^N = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$. We've shown that if $\mathbf{A}^N = \mathbf{0}$, the next lower power $\mathbf{A}^{N-1}$ is also **0**.

[45] Matrix multiplication isn't commutative, so we indeed need to multiply both sides by the same matrix *on the same side of the existing expression*.

But there's no limit to how many times we can repeat the same statement, such that $\mathbf{A}^{N-2} = \mathbf{0}$, $\mathbf{A}^{N-3} = \mathbf{0}$, all the way down to $\mathbf{A}^1 = \mathbf{0}$. The zero matrix isn't invertible, so we must have made a mistake in our logic somewhere. If our assumption that $\mathbf{A}^{-1}$ existed was correct, then since we were justified in every other step, we could not have reached this logical error. Therefore, it must not be possible for $\mathbf{A}^{-1}$ to exist.

Practice Set 3: Vector Spaces

Relevant Equations/Information

Vector Space: The notion of a vector space $\mathbb{V}$ is likely more abstract to you than earlier linear algebra concepts in this course. It is defined as a set of vectors *along with* the vector addition and scalar multiplication operators.[46] This entire set must satisfy 10 key properties as listed below.

1. Associativity of Vector Addition
2. Commutativity of Vector Addition
3. Additive Identity
4. Additive Inverse
5. Closure under Vector Addition
6. Associativity of Scalar Multiplication
7. Multiplicative Identity
8. Distributivity in Vector Addition
9. Distributivity in Scalar Addition
10. Closure under Scalar Multiplication

[46] These operators need not even be the addition and multiplication you're used to; we can literally redefine them entirely and still satisfy the vector space properties. If this confuses you, don't worry. It's not a point of emphasis in this class.

A couple of important examples of vector spaces you've already seen include $\mathbb{R}^N$, the vector space containing all n-dimensional vectors, and the set of all matrices $\mathbf{A}$ of a given size $n \times m$. Understanding the first of these examples is most critical for this class.

Basis: Given a vector space $\mathbb{V}$ and a set of vectors $\mathbb{S}$, the vectors in $\mathbb{S}$ constitute a basis for $\mathbb{V}$ if:

1. they are all linearly independent.
2. their span is $\mathbb{V}$.

This essentially means that a basis of $\mathbb{V}$ consists of the most condensed set of vectors possible that still span $\mathbb{V}$. It is a *minimal set of spanning vectors*. Consider what happens if we take a set of vectors known to be a basis, and add another vector to it. Does it remain a basis? It doesn't, because it is no longer a *minimal* set. We can remove the vector we just added and still have a linearly independent spanning set.

Similarly, suppose we have a linearly *dependent* spanning set of vectors; linear dependence implies redundancy, so once we strip out all the redundancy, the resulting set will still span the vector space but now will form a basis. Note that the vectors in the basis itself can vary; for example, considering the example of $\mathbb{R}^2$, there are an infinite number of pairs of vectors that form a basis. Consider $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ (standard basis), $\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, $\left(\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$, etc. To generalize, for $\mathbb{R}^N$, any N (and only N) *linearly independent* vectors form a basis!

Dimension: The dimension of $\mathbb{V}$ is the number of vectors in its basis, which is always the same regardless of our choice for the specific basis vectors.

Subspace: A subspace $\mathbb{U}$ of $\mathbb{V}$ is a subset of the vectors in $\mathbb{V}$ that satisfy the following:

1. Contains $\vec{0}$.
2. Closed under vector addition.
3. Closed under scalar multiplication.

Intuitively, a subspace is a *closed subset* of all the vectors in $\mathbb{V}$. Any linear combination of vectors in the subspace must also lie in that subspace. A couple other definitions related to subspaces (very analogous to how they're defined for vector spaces):

1. *Basis for a Subspace*: set of linearly independent vectors that span the subspace (*minimal set of subspace-spanning vectors*)
2. *Subspace Dimension*: number of vectors in subspace-basis.

Column Space and Row Space: The column space of a matrix $\underset{m \times n}{\mathbf{A}}$ is the span of the n columns in $\mathbf{A}$, and is also sometimes referred to as the range of $\mathbf{A}$. Similarly, the row space is the span of the n rows. Thinking about $\mathbf{A}$ as a linear transformation from $\mathbb{R}^N \to \mathbb{R}^M$, the column space is effectively the set of all outputs that this matrix can transform input vectors to. Note that in the general case, input vectors and output vectors can be different lengths; the column space describes all possible *output* vectors $\underset{m \times 1}{\vec{b}}$. It can be shown that range($\mathbf{A}$) forms a subspace of $\mathbb{R}^M$.[47]

Rank: The rank of $\mathbf{A}$ is defined as the dimension of the column space of $\mathbf{A}$ (aka `dim(range(A))` $\equiv$ `dim(col(A))`).[48] This is at most m, but certainly can be less, since an arbitrary $\underset{m \times n}{\mathbf{A}}$ is not guaranteed to have columns whose span is all of $\mathbb{R}^M$. Consider the simple counterexample of the zero matrix $\underset{m \times n}{\mathbf{0}}$, which maps all n-dimensional input vectors to the m-dimensional all-zero vector.

[47] Note that range($\mathbf{A}$) is not necessarily $\mathbb{R}^M$. This is explained in following definitions.

[48] It's all too easy to confuse an actual space consisting of vectors, like a matrix range describing the output (column) space, with the dimension of that space, which is just a single scalar number. Keep them straight!

In general, using the column-wise representation of matrix-vector multiplication as in eq. (8), we can show that rank(**A**) is the number of linearly independent columns in **A**. This is at most $\min(n, m)$. Why is this the case? Well, any output vector can be represented as a linear combination of the columns of **A**. But some of these columns might themselves be linear combinations of other columns, which means we can replace any redundant column with a weighted sum of the other columns. By removing all redundancies, we find that a matrix with $k(\leq \min(n, m))$ linearly independent column vectors can "unlock" exactly k dimensions in the output.

From these results, we find that rank(**A**) also equals the number of pivots in the RREF of **A**. Since each pivot must belong to a row *and* a column, the number of pivots in $\underset{m \times n}{\mathbf{A}}$ is limited by the smaller dimension. For a tall matrix ($m > n$), the columns are the limiting dimension; for a wide matrix ($n > m$), the rows are.

Null-Space: The null-space of a matrix $\underset{m \times n}{\mathbf{A}}$ is defined as the set of all $\vec{x}$ for which $\mathbf{A}\vec{x} = \vec{0}$; that is, the set of all inputs that get mapped to $\vec{0}$ (the zero output vector) by **A**. dim(null(**A**)) can be interpreted as the number of input directions for which the output is "compressed" down to zero.

Suppose you have a matrix and want to find out what dim(null(**A**)) is. It's equal to the number of free variables present in the RREF of **A**.

Procedure to Compute a Null-Space for a Given Matrix: Computing the null-space of **A** requires us to solve $\mathbf{A}\vec{x} = \vec{0}$. The procedure is as follows:

1. Put **A** in RREF. Initialize the set $S = \{\vec{0}\}$.
2. Check each column for leading entries and find the number of free (F) and basic (B) variables.
3. if $F = 0$, stop and skip to the last step.
4. if $F \neq 0$, repeat the following for each free variable:

   (a) Set that free variable to 1, and all others to zero.

   (b) Solve $\mathbf{A}\vec{x}$ under these conditions; add the solution vector to S.

5. Conclude that null(**A**) = span(S).

To practice applying the steps, let's do an example to find null(**A**):

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{1}{5} & 1 \\ 0 & 1 & \frac{7}{5} & 0 \end{bmatrix}$$

There are 2 free variables (x_3 and x_4), and 2 basic variables (x_1 and x_2). The

solutions are parameterized by the equations:

$$\begin{cases} x_1 \quad + \frac{1}{5}x_3 + x_4 = 0 \\ \quad x_2 + \frac{7}{5}x_3 \quad\quad = 0 \end{cases}$$

We set $x_3 = 1$, $x_4 = 0$. Solving $\mathbf{A}\vec{x}$, we find $\vec{x_1} = \begin{bmatrix} -\frac{1}{5} & -\frac{7}{5} & 1 & 0 \end{bmatrix}^\top$. Repeat with $x_3 = 0$, $x_4 = 1$; $\vec{x_2} = \begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}^\top$. Therefore:

$$\texttt{null(A)} = \text{span} \left\{ \begin{bmatrix} -\frac{1}{5} \\ -\frac{7}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Rank-Nullity Theorem: How is the number of free variables related to the total number of columns in a matrix? Well, each column of a matrix either contributes a "new direction" to the output or it is redundant with other columns and their already-discovered directions. In other words, each of n columns adds a dimension to `range(A)` or to `null(A)`. Therefore, the following holds:

$$\texttt{dim(range(A)) + dim(null(A))} = n \qquad (16)$$

Problems

✠ **Problem:** Is it *true* or *false* that the vectors in $\mathbb{S}$ below span $\mathbb{R}^2$?

$$\mathbb{S} = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$$

Answer: True

Quick Solution: We can form a plot as in fig. 6 and visually confirm that these vectors can indeed reach any point in $\mathbb{R}^2$. Also note that there are at least 2 unique directions represented by these vectors; we can conclude without much work that in this simple example, the given vectors span $\mathbb{R}^2$.

Long Solution: We appeal to the definition; can a linear combination of these vectors be used to create *any* vector $\vec{b}$ in $\mathbb{R}^2$?[49]

We could use an intuitive or visual approach, as in the QUICK SOLUTION, but this approach will quickly fall apart for larger dimensions. It's not so easy to plot a set of 4D vectors to confirm that they span $\mathbb{R}^4$! Let's take a more rigorous approach (or rather, 2 of them):

1. Eliminate Redundancy + Gaussian Elimination: In this approach, we first remove a redundant vector from the given set. Noticing $\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -3 \\ 1 \end{bmatrix} - 11\begin{bmatrix} -1 \\ 0 \end{bmatrix}$,
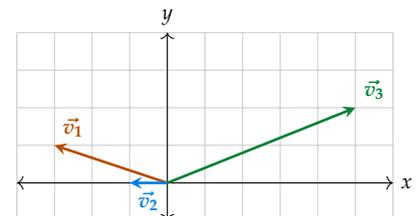
Figure 6: It is clear from the diagram that the given vectors span all of $\mathbb{R}^2$.

[49] Note that because there are 3 vectors that each have only 2 elements, they must be linearly dependent. This isn't a concern for span, but it's useful to get into the habit of noting these things for concepts like bases, subspaces, null-spaces, and more.

we eliminate the 3rd vector, leaving only the first two. Then, we have to mathematically formulate in words the question at the start: can a linear combination of these vectors create any arbitrary $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$? Using eq. (8), we must check if:

$$\alpha_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

has a unique solution in terms of b_1 and b_2. We use Gaussian Elimination on the augmented matrix below:

$$\begin{bmatrix} -3 & -1 & \bigg| & b_1 \\ 1 & 0 & \bigg| & b_2 \end{bmatrix}$$

Solving, we find that $\alpha_1 = b_2$, $\alpha_2 = -b_1 - 3b_2$. That is, there is a unique solution, so given any point in $\mathbb{R}^2$, we can scale the given vectors to reach that point. That is, we span $\mathbb{R}^2$.

2. We can also manually determine if the standard unit vectors $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in the span of the given vectors. If they are, then we can definitely create any other point in $\mathbb{R}^2$ using the standard unit vectors, so we will span $\mathbb{R}^2$.

So we check:

$$\hat{i} = -\begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad\qquad ✓$$
$$\hat{j} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad\qquad ✓$$

Indeed, we span $\mathbb{R}^2$.

There are many other ways to determine if a a set of vectors spans a space as well, and some approaches may be easier than other to apply in certain situations.

✠ **Problem:** Is it *true* or *false* that the vectors in S below form a basis for $\mathbb{R}^3$?

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 5 \end{bmatrix} \right\}$$

Answer: False

Quick Solution: We have 3 vectors in the candidate basis, but they are not linearly independent, and therefore cannot form a basis for $\mathbb{R}^3$.

Long Solution: Recall that a basis is a *minimal set of spanning vectors*. As a sanity check, we have exactly 3 vectors in a candidate basis for a 3-dimensional space, so we satisfy the criteria for being minimal. Now we need to check if all the vectors are linearly independent; if they are, then they span $\mathbb{R}^3$, so combining minimality with spanning, we can conclude that they form a basis.

At this point, we have many options to check linear independence of vectors; visual inspection is difficult for anything larger than 2D, so we opt to use Gaussian Elimination (forming an augmented matrix with each vector as a column and setting the solution vector to zero). Then, we solve! [50] Once we put the matrix in

[50] If this approach doesn't make sense, revisit eq. (9)! We're essentially trying to show that the only solution to $A\vec{x} = \vec{0}$ is the trivial solution, or that the null-space is trivial.

RREF form, we can determine `rank(A)`. Note that the rank gives the number of linearly independent columns; if the rank isn't 3, we can't have a basis!

$$\begin{bmatrix} 1 & 5 & -3 & 0 \\ 2 & -2 & 6 & 0 \\ 3 & 1 & 5 & 0 \end{bmatrix}$$

$$\underset{R_3-3R_1\to R_3}{\overset{R_2-2R_1\to R_2}{\Longrightarrow}} \begin{bmatrix} 1 & 5 & -3 & 0 \\ 0 & -12 & 12 & 0 \\ 0 & -14 & 14 & 0 \end{bmatrix}$$

$$\overset{\frac{7}{6}R_2-R_3\to R_3}{\Longrightarrow} \begin{bmatrix} 1 & 5 & -3 & 0 \\ 0 & -12 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

At this point, there are many ways to arrive at the same conclusion, and many ways to phrase it. We have a free variable ($\vec{c_3}$), so the vectors are linearly dependent. $\vec{c_3}$ is a linear combination of $\vec{c_1}, \vec{c_2}$, so our rank is only 2 (again, vectors are linearly dependent). `null(A)` (we just solved for the null-space by setting $A\vec{x} = \vec{0}$) is nontrivial.

The conclusion is, the vectors are not linearly independent so the vectors cannot form a basis for $\mathbb{R}^3$.

✠ **Problem:** The vectors $\vec{v_1}, \ldots, \vec{v_5}$ below are known to span $\mathbb{R}^3$ (feel free to confirm). Of the options listed, which set of vectors is a basis for $\mathbb{R}^3$?

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \vec{v_4} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \quad \vec{v_5} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

* $S_1 = \{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}\}$
* $S_2 = \{\vec{v_1}, \vec{v_3}, \vec{v_5}\}$
* $S_3 = \{\vec{v_1}, \vec{v_2}, \vec{v_4}\}$
* $S_4 = \{\vec{v_1}, \vec{v_3}, \vec{v_4}, \vec{v_5}\}$

Answer: S_2

Quick Solution: Immediately eliminate S_1 and S_4; by definition, a basis must have exactly as many vectors as the dimension of the vector space of interest (here, `dim(`$\mathbb{R}^3$`)=3`). With the remaining two sets, we can go through the exact same process as in the previous problem, performing Gaussian Elimination on the augmented matrix to find the set with linearly independent vectors. Doing so, we can find that the vectors in S_3 are linearly dependent, so only S_2 can be a basis.

✠ **Problem:** Suppose $\vec{v_1}$ and $\vec{v_2}$ are two vectors in set $\mathbb{W}$. If it is known that $\vec{v_1} + \vec{v_2}$ is not in $\mathbb{W}$, is $\mathbb{W}$ a subspace for any vector space?

Answer: No

Quick Solution: Note the properties of a subspace (especially relevant here is closure under vector addition), and recall that a subspace is a *closed* subset of a vector space; if the sum $\vec{v}_1 + \vec{v}_2$ is not in $\mathbb{W}$, $\mathbb{W}$ is not closed by definition.

✠ **Problem:** Performing Gaussian Elimination on $\mathbf{A}$ below yields $\mathbf{A}'$. Find a basis for $\texttt{col}(\mathbf{A})$.

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 4 & 3 \\ -1 & 2 & 1 & 2 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Answer: $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$

Quick Solution: We notice that $\vec{c}_1$ and $\vec{c}_3$ correspond to basic variables, whereas $\vec{c}_2$ and $\vec{c}_4$ do not contain leading ones in RREF form, so they correspond to free variables. Stated another way, $\vec{c}_2$ and $\vec{c}_4$ are linear combinations of $\vec{c}_1$ and $\vec{c}_3$. We can then take $\vec{c}_1$ and $\vec{c}_3$ from $\mathbf{A}$ and arrive at a basis for $\texttt{col}(\mathbf{A})$.

Long Solution: The tedious work has been done already in performing Gaussian Elimination; the task left is to interpret the results. Using the columns of $\mathbf{A}'$, we can quickly see which variables are basic, and the columns from $\mathbf{A}'$ corresponding to these basic variables allow us to form a basis for $\texttt{col}(\mathbf{A})$. [51] Here, we can see that $\vec{c}_2$ is a multiple of $\vec{c}_1$, and $\vec{c}_4$ is a linear combination of $\vec{c}_1$ and $\vec{c}_3$. That is, $\vec{c}_1$ and $\vec{c}_3$ form a basis for the column space. We now go back to $\mathbf{A}$ and select those columns and arrive at the answer as $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$.[52]

[51] Note that $\texttt{col}(\mathbf{A}) \equiv \texttt{col}(\mathbf{A}')$; that is, Gaussian elimination preserves the column space.

[52] Note that $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, the columns from $\mathbf{A}'$, also form a basis for $\texttt{col}(\mathbf{A})$. In the original question, this combination just wasn't one of the choices presented. It serves as a good reminder that bases aren't unique!

✠ **Problem:** Is it *true* or *false* for an $m \times n$ matrix $\mathbf{A}$ with pivots in every row, $\texttt{col}(\mathbf{A})=\mathbb{R}^M$?

Answer: True

Quick Solution: A pivot (leading entry) in a column means that column corresponds to a basic variable. If there are leading entries in all m rows, then the m columns containing those leading entries are all linearly independent.[53] The rank of a matrix is the number of linearly independent columns it has; here, this is m. The rank of a matrix also gives $\texttt{dim}(\texttt{col}(\mathbf{A}))$. From here, we conclude that $\texttt{col}(\mathbf{A})=\mathbb{R}^M$.

[53] Note that this also implies there are at least as many total columns as rows to hold these leading entries; $n > m$.

When $m < n$, the pivot in each of the m rows occupies a spot in one of the n columns, so the same result holds. Note that some of the n columns must be linearly dependent; they cannot each contain a pivot, because that would imply that there are at least as many rows as columns to hold the pivots (there can't be, since $m < n$).

✠ **Problem:** If we have $\underset{m \times n}{\mathbf{A}}$, what are the maximum values of $\texttt{dim}(\texttt{null}(\mathbf{A}))$ and $\texttt{dim}(\texttt{col}(\mathbf{A}))$?

Answer: $\underbrace{\texttt{dim(null(A))}}_{\text{max}} = n$, and $\underbrace{\texttt{dim(col(A))}}_{\text{max}} = \min(m, n)$

Quick Solution: For $\underbrace{\texttt{dim(null(A))}}_{\text{max}}$, consider the case of $\underset{m \times n}{\mathbf{0}}$. For $\underbrace{\texttt{dim(col(A))}}_{\text{max}}$, see the Rank part of the RELEVANT INFORMATION section for an explanation.

Long Solution: For the maximum $\texttt{dim(null(A))}$, consider $\underset{m \times n}{\mathbf{0}}$. Absolutely every input is mapped to 0 at the output, so the null-space, by definition, is the entire input space! $\texttt{dim(null(A))}$ is the number of input directions for which the output is zero; here, it's all of them ($= n$, since there are n columns). Another way to see this result is by invoking the Rank-Nullity Theorem, eq. (16). The zero matrix has rank 0, because it has no linearly independent columns. This implies that the $\texttt{dim(null(}\underset{m \times n}{\mathbf{0}}\texttt{))} = n$.

As for the maximum $\texttt{dim(col(A))}$, we can think about the number of pivots in the RREF of $\mathbf{A}$, as outlined in the Rank section. Alternatively, consider that any output can be represented as a linear combination of the linearly independent columns of $\mathbf{A}$, which is a most n. What if $m > n$? Do the additional rows "unlock" more output dimensions? Sadly not; in this case of a "tall" matrix, performing Gaussian Elimination will reveal that many equations (rows) are linearly dependent with other equations, so we encounter the same issue.[54]

[54] Notice the symmetry; thinking about a matrix in terms of its columns is not inherently different than thinking in terms of rows. This is one reason why $\texttt{dim(col(A))} = \texttt{dim(row(A))}$ for *any* matrix $\mathbf{A}$.

✠ **Problem:** Given $\mathbf{A}$ below, find $\texttt{dim(col(A))}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ -1 & 2 & 0 & 1 \\ 1 & -2 & 0 & -1 \\ 3 & 5 & 0 & 8 \end{bmatrix}$$

Answer: 2

Quick Solution: We perform Gaussian Elimination to find the number of linearly independent columns, and arrive at 2. Or we notice that $\vec{c}_3$ is $\vec{0}$ and adds nothing to our span, and $\vec{c}_4 = \vec{c}_2 + \vec{c}_1$, so $\vec{c}_4$ is redundant. Therefore, we have 2 linearly independent vectors, so $\texttt{rank(A)} \equiv \texttt{dim(col(A))} = 2$.

Long Solution: Let's go through and perform Gaussian Elimination for the practice:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ -1 & 2 & 0 & 1 \\ 1 & -2 & 0 & -1 \\ 3 & 5 & 0 & 8 \end{bmatrix}$$

$$\begin{array}{c} R_1+R_2 \to R_2 \\ R_3-R_1 \to R_3 \\ R_4-R_1 \to R_4 \end{array} \Longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & -4 & 0 & -4 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\xRightarrow[R_4 + \frac{1}{4}R_1 \to R_4]{R_3 + R_2 \to R_3} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From here, $\mathtt{dim(col(A))} = 2$.

✠ **Problem:** Is it *true* or *false* that a square matrix $\underset{n \times n}{\mathbf{A}}$ is invertible if and only if $\mathtt{rank(A)} = n$?

Answer: True

Long Solution: In many ways, this is a question about definitions. If $\mathtt{rank(A)} = n$, then all n columns are linearly independent. A matrix is only invertible if $\mathtt{dim(col(A))} = n$. Why? Because then, the output of the matrix has dimension n, and the input has dimension n by definition. We do not lose any dimensions going from the input to the output, so the matrix has a trivial null-space. Each input $\to$ output mapping is unique, so given an output, we can apply $\mathbf{A}^{-1}$ and find the corresponding input.

Another way to think about the concepts is to note that each linearly independent column will contain a leading entry in the RREF form of the matrix. Therefore, when we concatenate the identity matrix to $\mathbf{A}$ and reduce (go from eq. (14) to eq. (15)), we are certain we will actually reach identity on the left side (and therefore have the inverse on the right).

✠ **Problem:** Is it *true* or *false* that for a square matrix $\mathbf{A}$, $\mathtt{rank(A)} = \mathtt{rank(A^2)}$?

Answer: False

Quick Solution: We've seen this topic before; consider Nilpotent Matrices. There is no guarantee that applying a linear transformation twice preserves output vector directions (dimensions) in the same way as applying it once. It is also easy to generate examples to show this, such as $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.[55]

[55] It is worth noting that if $\mathbf{A}$ is full rank (invertible), then $\mathbf{A}^2$ is invertible, and is also therefore full rank. In this case, $\mathtt{rank(A)} = \mathtt{rank(A^2)}$. But the converse does not hold; for some noninvertible matrices, such as $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$, $\mathtt{rank(A)} = \mathtt{rank(A^2)} = \mathtt{rank(A^3)}$... so be careful!

Practice Set 4A: Page Rank, Eigenvalues and Eigenvectors

Relevant Equations/Information

Page Rank + Stability: Suppose that we can model the internet as a state diagram as in fig. 7, with each user belonging to a node (website) and having some chance of moving to a different site, or staying on the same one.[56] We start the system (or *initialize it*) with state of $\vec{x}[0]$ (a list of proportions). Over time, the proportions might "settle down" such that from one timestep to the next, we see little change, depending on the system behavior.

We may even reach an *equilibrium state* or *steady state*, denoted $\vec{x}^*$ and defined as a state for which applying the state transition matrix $\mathbf{P}$ does not change the state vector. Succinctly:

$$\vec{x}^* = \mathbf{P}\vec{x}^* \tag{17}$$

What's the point of finding $\vec{x}^*$? Well, the hope is that this vector gives information about the steady-state popularities of the websites (which can be used in generating advertisements, usage statistics, and more).

Let's consider the 2-node system in fig. 7. The associated state transition matrix (treating Google as node 1 and Bing as node 2) is $\mathbf{P} = \left[\begin{smallmatrix} 0.5 & 0.8 \\ 0.5 & 0.2 \end{smallmatrix}\right]$. Suppose we have 1000 people, and they each are equally likely to visit Google or Bing when they *first* go online. Our initial state is then $\vec{x}[0] = \left[\begin{smallmatrix} 500 \\ 500 \end{smallmatrix}\right]$. What happens after 1 timestep? Well, 50% of people on Google will move to Bing and the other half remain, so Bing gains 250 from Google. Similarly, 20% of Bing will move to Google and 80% remain, so Google gains 400 from Bing. Overall, Google gets 150 from Bing, so $\vec{x}[1] = \left[\begin{smallmatrix} 650 \\ 350 \end{smallmatrix}\right]$. From this, it seems like Google is more popular; but remember, this is only after one timestep! Things can change. Does our calculation of $\vec{x}[1]$ match the result of applying the matrix directly? We check:

$$\vec{x}[1] = \mathbf{P}\vec{x}[0] = \begin{bmatrix} 0.5 & 0.8 \\ 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} 500 \\ 500 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 650 \\ 350 \end{bmatrix}$$

[56] This logic is identical to that of the water pumps in previous notes; see fig. 3 and fig. 4 and possibly also the associated portion in RELEVANT INFORMATION for a refresher.
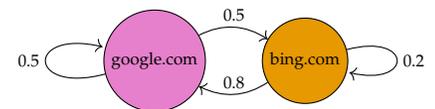
Figure 7: An absurdly simple Page Rank diagram with just 2 websites.

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.8 \\ 0.5 & 0.2 \end{bmatrix}$$
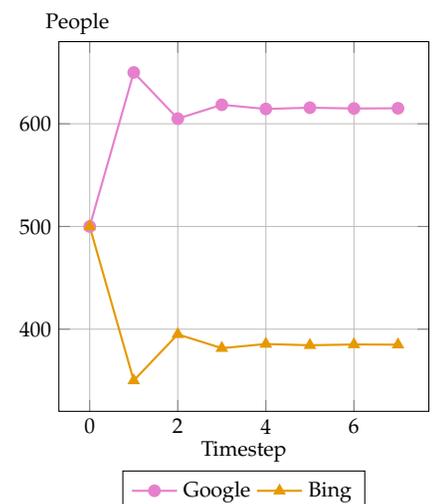
Figure 8: Distribution of people in each search engine over timesteps, corresponding to simple example in fig. 7.

Applying **P** gave the same result as our word-based calculation, as expected! Now, let's see what happens if we repeat this process many times; the results are plotted in fig. 8. Notice they seem to be approaching some values. But how can we rigorously show what they're approaching, and what are these values?

Eigenvalue/Eigenvector: What is an eigenvector? It represents a sort of stability point: vectors aligned with an eigenvector *will not change direction* under a linear transformation **A**. They will simply be scaled by some factor. The eigenvalue describes this stretching or compressing factor for vectors aligned with an eigenvector.[57] Because these two terms are so commonly used in conjunction, we often refer to an eigenvalue/eigenvector pair. We represent these ideas with the following definition: $\vec{x}$ is an eigenvector of **A** with associated eigenvalue λ if:

$$\mathbf{A}\vec{x} = \lambda\vec{x} \tag{18}$$

[57] Note that eigenvectors are a property of the matrix itself and do not depend on the specific vector being transformed.

The concept is best illustrated with some diagrams. Suppose we have $\mathbf{A} = \begin{bmatrix} 0.9 & 1 \\ 0.7 & 1.2 \end{bmatrix}$. This matrix has an eigenvector/eigenvalue pair $(\vec{v}_1, \lambda_1) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 1.9\right)$. This means any vector that's a multiple of the eigenvector, when it's transformed by **A**, will become a scaled version of itself. Let's take $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Does eq. (18) hold (does $\mathbf{A}\vec{x} = 1.9\vec{x}$)? Indeed it does! We can verify this computationally as $\begin{bmatrix} 0.9 & 1 \\ 0.7 & 1.2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 3.8 \\ 3.8 \end{bmatrix}$, and illustrate it in fig. 9. Note that scaling a given eigenvector for an eigenvalue will still produce a valid eigenvector, since the vector's direction will not be changed.

Now let's go back to the original example. Suppose we're already many timesteps into the dynamics of the system, and we have $\vec{s}$ and $\vec{s'}$ (see in fig. 10) before and after one more timestep, in which the transformation **P** from above is applied. $\vec{v}$ is a relevant eigenvector with $\lambda = 1$. Note how $\vec{s}$ and $\vec{s'}$ are related; they're the same! What does this mean for the Page Rank example? Well, if the state vector ever becomes $\vec{x}[i] = \begin{bmatrix} \frac{8}{13} \cdot 1000 \\ \frac{5}{13} \cdot 1000 \end{bmatrix} \approx \begin{bmatrix} 615.4 \\ 384.6 \end{bmatrix}$, then those proportions will not change; they are aligned with the eigenvector $\vec{v}_1$ which scales input vectors by exactly 1. We see this steady-state pattern in fig. 8!

For the same **P**, when the initial state $\vec{x}[0]$ is transformed, it becomes $\vec{x}[1]$. Note that $\vec{x}[0]$ is *not* aligned with the eigenvector, so it doesn't just get scaled; its direction changes.

Determinant: The determinant is a quantity we can define for any square matrix.

Figure 9: $\vec{x}$ is transformed into a $\lambda_1\vec{x} = \vec{x'}$ by **A** since it's aligned with $\vec{v}_1$ of **A**.

$$\mathbf{A} = \begin{bmatrix} 0.9 & 1 \\ 0.7 & 1.2 \end{bmatrix}$$
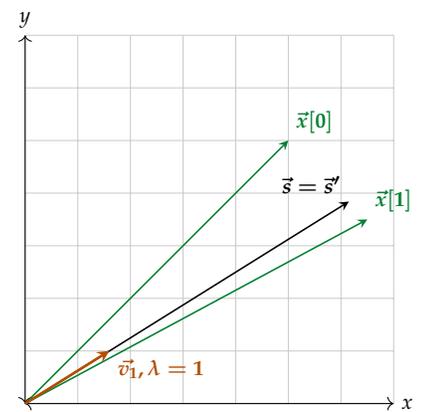
Figure 10: Notice how $\vec{x}[0]$ changes into $\vec{x}[1]$, but $\vec{s}$ and $\vec{s'}$ are the same, both aligned with $\vec{v}_1$. Everything except $\vec{v}_1$ is scaled down by a factor of 100 for visualizing purposes. But remember, the direction is what matters.

For a 2×2 matrix, the formula is:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc \tag{19}$$

We can calculate determinants for higher dimension matrices as well, as described here. There is a nice geometrical intuition for the determinant; for a 2×2 matrix, if we treat each column vector of a square matrix as one leg of a parallelogram, the determinant gives the area of of the parallelogram. We take the same $\mathbf{A} = \begin{bmatrix} 0.9 & 1 \\ 0.7 & 1.2 \end{bmatrix}$ to illustrate in fig. 11. According to eq. (19), the determinant is $0.9 \cdot 1.2 - 0.7 \cdot 1 = 0.38$. If we calculate the area of the parallelogram defined by the two column vectors, we also get 0.38! For a 3×3 matrix, the shape is a parallelipiped, and we form hyper-volumes in higher dimensions.

On the other hand, in fig. 12, we notice what happens if the column vectors are linearly dependent, as for $\mathbf{B} = \begin{bmatrix} 0.4 & 0.9 \\ 0.8 & 1.8 \end{bmatrix}$; they will lie on top of each other, so the area enclosed will be zero! Similarly in 3D, if any column vectors are multiples of each other, we "squash" a volume into a plane or a line. A useful general result to remember is that the *determinant of any square matrix is zero if the columns are linearly dependent*.

Computing Eigenvalues and Eigenvectors: One major question remains from earlier; given a matrix, how do we find the $(\vec{v}, \lambda)$ pairs? How many are there? What does it mean if two vectors have the same eigenvalue, or if there are two of the same vector? Let's apply our newfound knowledge about determinants!

We can put eq. (18) in the following form:

$$(\mathbf{A} - \lambda \mathbf{I}_n)\, \vec{x} = \vec{0} \tag{20}$$

Solving this equation for nonzero (nontrivial) solutions $\vec{x}$ will yield our eigenvectors! There's only 1 parameter here that can vary in this equation: λ. Note that $\mathbf{A}$ and $\mathbf{I}_n$ are fixed. The form of the above equation makes this clear, but we are forming a new matrix $\mathbf{A}' = (\mathbf{A} - \lambda \mathbf{I}_n)$. Note the generalized form of $\mathbf{A}'$:

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} - \lambda \end{bmatrix} \tag{21}$$

We want to find values of λ that cause the determinant of $\mathbf{A}'$ to become zero (linear
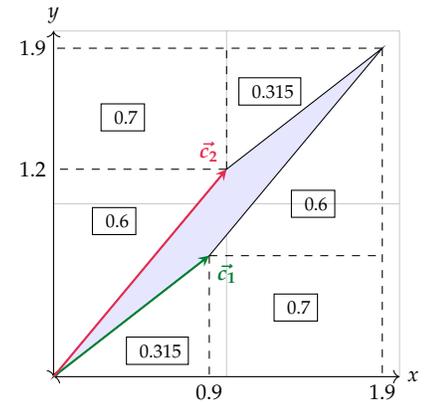
Figure 11: Plot of the columns of $\mathbf{A}$ from earlier. `det(A)` = 0.38, which is the same as the area of the labeled parallelogram. This area can be found by forming the square from $(0,0)$ to $(1.9, 1.9)$ and subtracting out the areas of various triangles/rectangles (the boxed values), as shown. $1.9^2 = 3.61$. $3.61 - 2(0.7) - 2(0.6) - 2(0.315) \overset{\checkmark}{=} 0.38$.

$$\mathbf{A} = \begin{bmatrix} 0.9 & 1 \\ 0.7 & 1.2 \end{bmatrix}$$
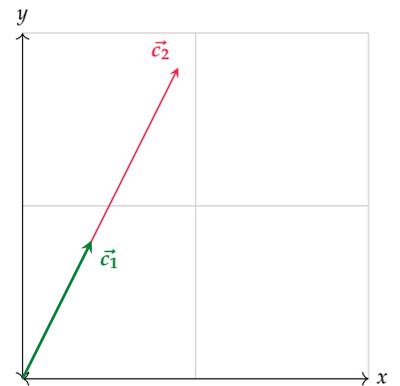
Figure 12: Plot of the linearly dependent columns of $\mathbf{B}$. `det(B)` $= 0$, since the columns are overlapping (aligned) and the area between them is zero.

$$\mathbf{B} = \begin{bmatrix} 0.4 & 0.9 \\ 0.8 & 1.8 \end{bmatrix}$$

dependence of the columns creates a nontrivial null-space for $\mathbf{A}'$). The determinant of $\mathbf{A}'$ (in the case of a 2×2) is given by eq. (22) below, and solving this quadratic equation, we can find the λ values. This part on the left is called the *characteristic polynomial*.

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \tag{22}$$

Then, for each λ_i we find, we revisit eq. (20) and solve for the corresponding $\vec{x}_i$. See the first two problems and explanations for a simple example.

If a matrix has repeated eigenvalues, they may (or may not) have distinct eigenvectors. If the eigenvectors are distinct, they form an eigenspace. The term may seem daunting, but it's exactly like the concept of span; the eigenspace is a subspace, the span of all eigenvectors for that eigenvalue (note that this includes $\vec{0}$). Any input vectors that lie in this space (for 2 distinct eigenvectors, a plane) will be scaled by the shared eigenvalue under a linear transformation. Complex eigenvalues can exist as well; they are much harder to visually understand, but mathematically, we find them using the exact same process as before.

Computing and Interpreting the Steady State: We introduced steady-states and stability conceptually earlier, but now we are equipped to show how to find a steady-state $\vec{x}^*$. At this point, we have a definition for $\vec{x}^*$ given in eq. (17). Let's pattern match to eq. (18); by definition, the steady-state eigenvalue is 1. Therefore, to solve for the steady-state of a system represented by $\mathbf{P}$, we solve eq. (20), substituting $\lambda = 1$.[58]

Predicting System Behavior for General Initial States: Given a system and an initial state, can we predict how it'll dynamically change over time? We saw in the Page Rank example that we seem to approach a sort of steady-state after many timesteps, but under what conditions does this happen?

Simpler Case, $\vec{x}[0] = \alpha\vec{v}$: Suppose our initial state is actually a perfect multiple of an eigenvector of the system. Over time, upon repeated applications of $\mathbf{A}$, we accumulate factors of λ; ultimately, $\mathbf{A}^n\vec{x} = \alpha\left(\lambda^n\vec{x}\right)$.[59] Based on this pattern, we notice the following behaviors based on the value of λ as $n \to \infty$:

- $\lambda > 1$: $\vec{x}[n] \to \infty$ (exponential growth).

- $\lambda = 1$: $\vec{x}[n] \to k\vec{v}$ (steady-state).

- $0 < \lambda < 1$: $\vec{x}[n] \to \vec{0}$ (exponential decay).

[58] Note that this amounts to solving for the null-space of $(\mathbf{P} - \mathbf{I_n})$; see the previous RELEVANT INFORMATION section for a refresher on the process.

[59] Note how this is derived:

$$\vec{x}[0] = \alpha \cdot \vec{v}$$
$$\vec{x}[1] = A\vec{x}[0] = \alpha\lambda\vec{v}$$
$$\vec{x}[2] = A\vec{x}[1] = \alpha\lambda^2\vec{v}$$
$$\vec{x}[3] = A\vec{x}[2] = \alpha\lambda^3\vec{v}$$
$$\vdots$$

- $\lambda = 0$: $\vec{x}[n] = 0\vec{v} = \vec{0}$ (instantaneous disappearance).

- $\lambda < 0$: Take $|\lambda|$ and refer to the appropriate case above. But recognize the sign switches at each timestep.

General Case, $\vec{x}[0] = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \ldots + \alpha_n \vec{v_n}$. Recognize this form? It says that the initial state is now not a scalar multiple of just one eigenvectors, it's a linear combination of all of them![60] But this case devolves into the previous one; we can simply treat each element individually, apply the techniques from the *Simpler Case*, and put them back together. The final form is as follows:

$$\vec{x}[n] = \alpha_1 \left(\lambda_1^n \vec{v_1}\right) + \alpha_2 \left(\lambda_2^n \vec{v_2}\right) + \ldots + \alpha_n \left(\lambda_n^n \vec{v_n}\right) \tag{23}$$

Given a matrix $\mathbf{A}$ and some initial state $\vec{x}$, how can we actually get to this equation format? First, we solve for the $(\vec{v_i}, \lambda_i)$ pairs of $\mathbf{A}$. Then, we use Gaussian elimination to find the α_i's; Putting eq. (23) in matrix form yields:

$$\vec{x}[0] = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

and we compute the inverse of the matrix of eigenvectors, arriving at:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}^{-1} \vec{x}[0] \tag{24}$$

Now, let's see how to interpret all this! Let's assume that we drop any terms corresponding to $\alpha_i = 0$ since those terms are, well, zero. Then, with what remains, we can make some intuitive observations:

1. (diverge) Any $|\lambda_i| > 1$: $\vec{x}[n] \to \infty$. Even if other components in the linear combination decay, the state itself "blows up" to ∞ as this component overshadows all others.

2. (diverge) Any $|\lambda_i| = -1$: $\vec{x}[n] \to$? because that component oscillates forever.

3. (converge) All $-1 < \lambda_i \leq 1$: $\vec{x}[n] \to \vec{x}^*$, steady-state! Each ith term either decays to zero ($|\lambda_i| \leq 1$) or stays the same ($|\lambda_i| = 1$), such that $\vec{x}^* = \sum_{(i,\lambda_i=1)} \alpha_i \vec{v_i}$. We can normalize this steady-state if we want proportions (column values sum to 1) rather than absolute numbers (such as $\vec{x} = \begin{bmatrix} 615.4 \\ 384.6 \end{bmatrix}$ from the Page Rank example before).

[60] Note that this is still not *fully general*; we assume here that all initial states are in the span of the eigenvectors of $\mathbf{A}$, which isn't guaranteed.

Some Useful Information:

- If a matrix has n distinct real eigenvalues, their n associated eigenvectors are all linearly independent. See course notes for a proof!
- An eigenspace for a given eigenvalue is the span of all eigenvectors, including $\vec{0}$, and is also a subspace by definition.
- Say we calculate an eigenvector for an eigenvalue; we can pick any scalar multiple of the result and this will still be an eigenvector, since scaling a vector does not change its direction. This follows from the scalar multiplication property of subspaces.
- A given eigenvector can only be associated with one eigenvalue, since a vector can only be scaled by some single value upon being transformed by a matrix. But, an eigenvalue can be associated with multiple eigenvectors, the span of which form an eigenspace.
- If a matrix has some $\lambda = 0$, then for some $\vec{x}$, $\mathbf{A}\vec{x} = \lambda\vec{x} = \vec{0}$, so $\mathbf{A}$ has a nontrivial null-space. Therefore, it is not invertible.
- If a matrix has some $\lambda = 1$, then any initial state that is aligned with the corresponding eigenvector is a steady-state. More generally, any initial state for which $\lambda = 1$ comprises part of the linear combination *potentially* has a nonzero steady-state, so long as all other $|\lambda_i| < 1$.
- The rotation matrix (that rotates any vector by θ degrees counterclockwise) is:

$$\mathbf{A_R} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

For example, confirm that $(\mathbf{A_{R,90°}}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Problems

✠ **Problem:** What are the eigenvalues of $\mathbf{A}$?

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Answer: $-1, 5$

Quick Solution: The characteristic polynomial is $(3 - \lambda)(1 - \lambda) - 8 = 0$. Solving, $\lambda_1 = -1, \lambda_2 = 5$.

Long Solution: As promised, this is the problem to showcase many of the ideas mentioned in the RELEVANT INFORMATION section. To find the eigenvalues, we need to find the values of $\lambda_{1,2}$ that make eq. (18) hold. Rearranging, we arrive at

eq. (20). What is this equation when it's expanded?

$$\left(\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 - \lambda & 2 \\ 4 & 1 - \lambda \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We need to set $\det\left(\begin{bmatrix} 3 - \lambda & 2 \\ 4 & 1 - \lambda \end{bmatrix} \right) = 0$ because by definition, the columns will then be linearly dependent and we will have a nontrivial null-space, so we can solve for actual nonzero vectors $\vec{x}$ that make the equation true. Noting eq. (19) (or more directly, eq. (22)) we arrive at $(3 - \lambda)(1 - \lambda) - 4 \cdot 2 = 0$. From here, we find that $\lambda^2 - 4\lambda - 5 = 0$, and $\lambda_{1,2} = \frac{4 \pm \sqrt{16 - (4 \cdot -5)}}{2} = 2 \pm 3 = -1$ and 5.

✠ **Problem:** Extending the previous problem, given the same **A**, what are the eigenvectors?

Answer:

$$(\vec{v_1}, \lambda_1) = \left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}, -1 \right) \equiv \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}, -1 \right)$$

$$(\vec{v_2}, \lambda_2) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 5 \right)$$

Quick Solution: Using $\lambda_1 = -1$ and $\lambda_2 = 5$, we substitute each λ_i into eq. (20) and solve the null-space of the resulting modified matrix. For a given eigenvalue, the resulting null-space constitutes the corresponding eigenvector (or perhaps eigenspace, though here it's just 1 vector per eigenvalue). That's how we arrive at the ANSWER above.

Long Solution: From before, we already have the eigenvalues: $\lambda_1 = -1, \lambda_2 = 5$. Now, we need to solve for the corresponding eigenvectors. Let's take each individually:

$\lambda_1 = -1$: We plug -1 into eq. (20) and arrive at:

$$\left(\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \right) \vec{x} = \vec{0}$$

We need to solve for $\text{null}\left(\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \right)$. Applying the techniques in the previous PRACTICE SET, we put the matrix into RREF:

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

x_2 is a free variable since the second column has no pivots. The solutions are parameterized by $x_1 + \frac{1}{2}x_2 = 0$. We set $x_2 = 1$ and solve $\mathbf{A}\vec{x} = \vec{0}$ to get $\vec{v_1} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$.

Therefore:

$$(\vec{v_1}, \lambda_1) = \left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}, -1 \right) \equiv \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}, -1 \right)$$

.

$\lambda_1 = 5$: We plug 5 into eq. (20) and arrive at:

$$\left(\begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \right) \vec{x} = \vec{0}$$

To solve for $\text{null} \left(\begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \stackrel{\text{RREF}}{\Longrightarrow} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right)$, we again set $x_2 = 1$ and solve:

$\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$(\vec{v_2}, \lambda_2) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 5 \right)$$

For eigenvectors, the direction is what's important so we can scale it arbitrarily to match the multiple-choice options given in the original question.

✠ **Problem:** Is it *true* or *false* that a noninvertible $\underset{n \times n}{\mathbf{A}}$ has some $\lambda_i = 0$?

Answer: True

Quick Solution: We can make the following argument: If the matrix is not invertible, it has a nontrivial null-space.[61] Then, by definition, there is some nonzero $\vec{x}$ for which $\mathbf{A}\vec{x} = \vec{0}$. We pattern match to eq. (18) and notice the equation is exactly the same if we multiply the right by $\vec{x}$: $\mathbf{A}\vec{x} = 0\vec{x}$. This is kind of like pulling a scalar 0 out of $\vec{0}$, leaving $0\vec{x}$. Now, we clearly see $\lambda = 0$.

[61] The second bullet under the *Some Useful Information* section goes in the "other direction," showing that if a matrix has $\lambda = 0$, it is noninvertible. This question is almost identical.

✠ **Problem:** Is it *true* or *false* that for an invertible $\mathbf{A}$ with some λ, $\mathbf{A}^{-1}$ has eigenvalue $\frac{1}{\lambda}$?

Answer: True

Quick Solution: Given invertibility, all $\lambda_i \neq 0$ (see previous question or the *Some Useful Information* section. We therefore need not worry about dividing by zero in creating $\frac{1}{\lambda}$). We can start at eq. (18): left-multiply both sides by $\mathbf{A}^{-1}$ to get $\vec{v} = \mathbf{A}^{-1}\lambda\vec{v} \implies \mathbf{A}^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$. Pattern match to eq. (18) again and we've shown the statement is true!

✠ **Problem:** Students A and B calculate $\vec{v}$ for a given λ: $\vec{v_A}$ and $\vec{v_B}$ are as given below. Since the two are not the same, A says that B must have made a mistake, but B reassures A that both are valid. Who should you trust?

$$\vec{v_a} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \quad \vec{v_b} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 2 \end{bmatrix}$$

Answer: B

Quick Solution: The key steps here are as follows. First, recognize that $\vec{v}_A$ is just a 2x-scaled version of $\vec{v}_B$, and that they therefore represent the same direction. Second, understand that this direction is the most relevant information communicated by an eigenvector. It's important to understand that the eigenspace is a subspace, so if the vectors are related by a scalar, they must both lie in the eigenspace of λ. The direction is same for both students' vectors, so B is correct.[62]

[62] Let's assume they aren't both wrong :-)

✠ **Problem:** Is it *true* or *false* that $\mathbf{A}\vec{x}_1 = \lambda_1\vec{x}_1$ and $\mathbf{A}\vec{x}_1 = \lambda_2\vec{x}_1 \implies \lambda_1 = \lambda_2$?

Answer: True

Quick Solution: The crux of the question is, can the same eigenvector $\vec{x}_1$ have distinct associated eigenvalues λ_1 and λ_2? Thinking about a physical diagram may help clarify; $\lambda \neq \lambda_2$ would require some single initial state or vector aligned with $\vec{x}_1$ to be scaled by two *different* values upon being transformed by $\mathbf{A}$! This cannot happen, so the two eigenvalues cannot be distinct; the statement is true.

✠ **Problem:** Is it *true* or *false* that $\mathbf{A}$ with only real entries can have complex eigenvalues?

Answer: True

Quick Solution: Eigenvalues of a matrix are determined by the solutions to the characteristic polynomial (eq. (22) for a 2×2). Quadratic equations have complex roots whenever the term in the square root is negative. In other words, given $\underset{2\times2}{\mathbf{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $(a+d)^2 - 4(ad - bc) < 0$, the eigenvalues will be complex. This can certainly happen without complex matrix entries; consider $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -5 & -1 \end{bmatrix}$ where $\lambda_{1,2} = \pm i\sqrt{14}$.

✠ **Problem:** Is it *true* or *false* that if $\mathbf{A}^\top$ has eigenvalue λ, $\mathbf{A}$ also has eigenvalue λ?

Answer: True

Quick Solution: $\det(\mathbf{A} - \lambda\mathbf{I_n}) = \det(\mathbf{A}^\top - \lambda\mathbf{I_n})$ since a matrix and its transpose have the same determinant. λ is therefore an eigenvalue of both $\mathbf{A}$ and $\mathbf{A}^\top$ by definition.

Long Solution: How do we derive the eigenvalues for $\mathbf{A}$? We start at eq. (20) and set $\det(\mathbf{A} - \lambda\mathbf{I_n}) = 0$. We have to somehow introduce an $\mathbf{A}^\top$ term in this equation, but we can't apply it individually to $\mathbf{A}$. Instead, we use the fact that the transpose is a linear operator, and distribute it as follows: $(\mathbf{A} - \lambda\mathbf{I_n})^\top = (\mathbf{A}^\top - (\lambda\mathbf{I_n})^\top)$. But note that $\mathbf{I_n}^\top = \mathbf{I_n}$ so $(\mathbf{A}^\top - (\lambda\mathbf{I_n})^\top) \equiv (\mathbf{A}^\top - \lambda\mathbf{I_n})$.

Now, we equate and take determinants: $\det(\mathbf{A} - \lambda\mathbf{I_n}) = \det(\mathbf{A}^\top - \lambda\mathbf{I_n})$. Why can we equate these expressions? It's because a matrix and its transpose have the same determinant.[63] Note that now, by definition, λ is an eigenvalue of both

[63] Intuitively, See this article which relies on the idea of elementary matrices (not covered) or this more advanced, out-of-scope stackoverflow post that deals with 16B-level topics (and beyond, for some answers).

$\mathbf{A}$ and $\mathbf{A}^\top$. Notice that this statement can also be interpreted as a proof: Prove that if $\mathbf{A}$ has some eigenvalue λ, $\mathbf{A}^\top$ have that same eigenvalue. This gives a concrete start point and end point to work with. It provides motivation for taking the transpose of the determinant term in the first place (previously we had no equations involving $\mathbf{A}^\top$, but now we do!) So we can then begin manipulating and rearranging equations as we did above.

✠ **Problem:** Given noninvertible $\mathbf{A}_{n\times n}$, what are the minimum and maximum number of eigenvalues?

Answer: Min: 1, Max: n

Quick Solution: Given noninvertibility, some $\lambda_i = 0$, so we have at least 1 eigenvalue. Indeed, all eigenvalues can be 0 (such as for $\mathbf{0}$)! However, noninvertibility does not place any other restrictions on our set of eigenvalues, so all other eigenvalues can be distinct from this λ_i. As an example, consider $\mathbf{A}$ below which has $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 3$. Therefore, our maximum is n.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

✠ **Problem:** Given the state transition diagram below, find the steady-state vector with $\vec{x}[t]$ and $\vec{x}[0]$ defined below. If there is no steady-state, show why.
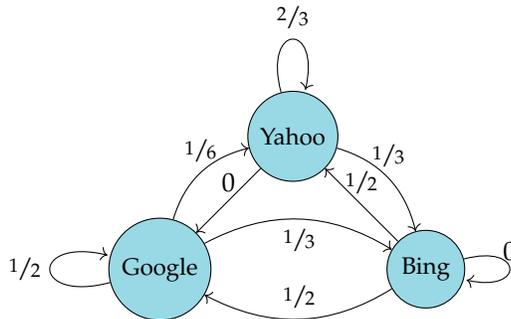
$$\vec{x}[t] = \begin{bmatrix} x_{\text{google}}[t] \\ x_{\text{yahoo}}[t] \\ x_{\text{bing}}[t] \end{bmatrix} \qquad \vec{x}[0] = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$$

Answer: $\vec{x}^* = \begin{bmatrix} 75 \\ 150 \\ 75 \end{bmatrix}$

Quick Solution: The general procedure is as follows: check if the system has a

steady-state by creating the state transition matrix and checking if it has $\lambda = 1$, by plugging $\lambda = 1$ into eq. (20) and checking if the determinant is zero. This is identical to computing $\texttt{null}(\mathbf{A} - \mathbf{I_3})$ and seeing if it's nontrivial, with the added benefit of actually figuring out what the steady-state is. Once we solve for the null-space (here, a single vector, since there's one free variable in the system) and re-scale to account for the system being conservative and the initial state having 300 people, we arrive at $\vec{x}^* = \begin{bmatrix} 75 \\ 150 \\ 75 \end{bmatrix}$.

Long Solution: The first step is to establish whether we even have a steady-state, and we can do so by figuring out if $\lambda = 1$ is an eigenvalue of the state transition matrix. We have to generate this from the diagram's values:

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

While one might be tempted to solve for all of the $(\vec{v}, \lambda)$ pairs of $\mathbf{A}$ first, this is actually unnecessary since our problem-solving goal is more narrowly defined.[64]

[64] Well, that and the fact that the formula for the determinant of anything higher dimension than a 2×2 is out of scope.

So how do we check if 1 is an eigenvalue? We substitute into eq. (20) and see if the equation holds!

$$\texttt{det}(\mathbf{A} - \lambda \mathbf{I_3}) = 0$$

$$\texttt{det}\left(\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

$$\texttt{det}\left(\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix} \right) = 0$$

We don't actually know how to compute this directly,[65] but we do know how the determinant of a matrix relates to its null-space. Here, we find $\texttt{null}(\mathbf{A} - \mathbf{I_3})$, and if it's nontrivial, then that's our steady-state! Let's begin:

[65] If you do though, you'll find the equation is true, so there's a steady-state. But this still doesn't answer the full question of what the steady-state actually is. However, it can be a good conceptual check before going through the full row-reduction of $\mathbf{A}$ only to find the null-space is trivial.

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix} \xRightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

We notice that there's a free variable corresponding to column 3, x_{bing}. We can parameterize the solutions:

$$x_{\text{google}} = x_{\text{bing}}$$

$$x_{\text{yahoo}} = 2x_{\text{bing}}$$

Setting $x_{\text{bing}} = 1$, we find that $x_{\text{google}} = 2$, $x_{\text{yahoo}} = 1$. So our steady-state vector has the form $\vec{x}^* = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Are we done? Well, the system is conservative

(based on the column sums of the state transition matrix) so we should retain the same number of people as in the initial state. What we've found so far are the *relative proportions* of each search engine's share of total users in steady-state. There are 300 total people but our current $\vec{x}^*$ only has 4, so we rescale $\vec{x}^*$ by multplying each component by 75 (to make the column sum 300): $\vec{x}^* = \begin{bmatrix} 75 \\ 150 \\ 75 \end{bmatrix}$.

� **Problem:** Find the matrix $\underset{2\times2}{\mathbf{A}}$ that reflects vectors in $\mathbb{R}^2$ about the line through the origin in the direction of $\vec{v}_r$ below. See fig. 13 for a visual aid.

Answer: $\begin{bmatrix} 15/39 & 12/13 \\ 12/13 & -15/39 \end{bmatrix}$

Quick Solution: This question requires a couple realizations; there is little symmetry to the given vector, so most directions (such as $\vec{x_2} = \begin{bmatrix} 3 & 0 \end{bmatrix}^\top$ in fig. 13) will end up at unintuitive points ($\vec{x_{2,r}}$) that don't help us solve the problem. But there are two useful directions; exactly aligned with and exactly perpendicular to $\vec{v}_r$. These have predictable, "nice"-valued reflected vectors. So we can use that to set up the following matrix-vector equations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

We can also form a singular, more directly solvable matrix-vector equation that combines both of the above sets of equations. This way, it might be easier to apply a standard solving technique like Gaussian Elimination. We write this combined equation as follows:

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -3 \end{bmatrix}$$

We can solve either of these forms for the variables of interest (a_{11}, a_{12}, a_{21}, a_{22}) and once we put them back into the original matrix form, we arrive at:

$$\mathbf{A} = \begin{bmatrix} 15/39 & 12/13 \\ 12/13 & -15/39 \end{bmatrix} \equiv \begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix}$$

Long Solution: A natural followup question is, why is this question in this practice set? What's the connection to eigen-stuff? Well, note that we could have solved this question another way after the initial realization.[66] Instead of setting up a system of equations to solve directly, we could also have seen that 1 and -1 must be eigenvalues of the matrix we're looking for. The scaling factor for vectors
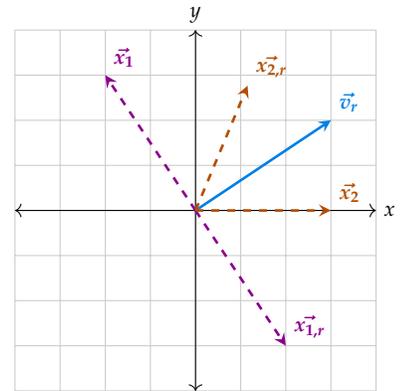
Figure 13: Diagram showing how various (dashed) vectors should be reflected about $\begin{bmatrix} 3 & 2 \end{bmatrix}^\top$ (the solid vector).

[66] If this feels a lot longer, that's because it is. In most situations for small cases, you'd likely just solve the system in the QUICK SOLUTION. But the concepts here are worth understanding; the steps are here for completeness.

aligned with $\vec{v}_r$ is 1, so one of our pairs is $\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}, 1\right)$. Similarly, the other pair is $\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}, -1\right)$. Using these and eq. (20), we construct the following:

$$(\mathbf{A} - \lambda_1 \mathbf{I_2})\, \vec{x} = \vec{0}$$

$$\begin{bmatrix} a_{11} - 1 & a_{12} \\ a_{21} & a_{22} - 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 3a_{11} - 3 + 2a_{12} \\ 3a_{21} + 2a_{22} - 2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 3a_{11} + 2a_{12} \\ 3a_{21} + 2a_{22} \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 3a_{11} + 2a_{12} \\ 3a_{21} + 2a_{22} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Now, for the next pair:

$$(\mathbf{A} - \lambda_2 \mathbf{I_2})\, \vec{x} = \vec{0}$$

$$\begin{bmatrix} a_{11} + 1 & a_{12} \\ a_{21} & a_{22} + 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} -2a_{11} - 2 + 3_{12} \\ -2a_{21} + 3a_{22} + 3 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} -2a_{11} + 3a_{12} \\ -2a_{21} + 3a_{22} \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} -2a_{11} + 3a_{12} \\ -2a_{21} + 3a_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

It's the exact same as before if we expand the equations out! But now, we've made the connection to eigen-concepts more explicit.

Practice Set 4B: Change of Basis

Relevant Equations/Information

Defining a Change of Basis: Previously we've seen that a basis for a vector space is a minimal spanning set of vectors. We can also define the *standard basis* vectors. For example, the standard basis for $\mathbb{R}^3$ is the set $\mathbb{E}$ below:

$$\mathbb{E} = \left(\hat{i}, \hat{j}, \hat{k} \right)$$
$$\equiv (\vec{e_1}, \vec{e_2}, \vec{e_3})$$
$$\equiv \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Take $\vec{x} = \begin{bmatrix} 1 & 3 & -4 \end{bmatrix}^\top$. We can represent this vector as a linear combination of the standard 3D basis vectors: $\vec{e_1} + 3\vec{e_2} - 4\vec{e_3}$. Let's go through a couple examples using a simpler 2D vector $\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. For later, let's define the elements of $\vec{v}$ to be v_1 and v_2.

fig. 14 shows how we plot $\vec{v}$. Simple, right? The point is to recognize that if you're asked to plot a vector, the basis is normally implicit as the standard basis vectors, so you'd have generated the same plot. But what if instead of using the standard basis, you actually had:

$$\mathbb{E}' = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \right)$$

Then your *perspective* is different because your defining grid has changed! To draw the new vector:

$$2\vec{e_1}' + 5\vec{e_2}' = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

This means that if we preserve the coordinates of the vector, the physical vector changes from $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ -0.5 \end{bmatrix}$, as in fig. 15. But oftentimes, we actually want to do the

$\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{v}$

$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
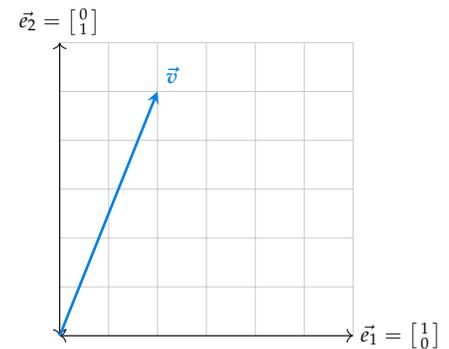
Figure 14: Standard Basis: a simple 2D vector, making explicit how it's a linear combination of the standard 2D basis vectors.

opposite: preserve the geometrical vector but modify its coordinates such that when plotted in the new basis, the original and final vector are physically the same. This is what's commonly called a *change of basis operation* on a vector. We are re-expressing the coordinates of $\vec{v}$ in the new basis $\mathbb{E}'$.

This requires solving a simple system of equations: we know that the basis vectors define the linear transformation matrix:

$$\mathbf{A} = \begin{bmatrix} | & | \\ \vec{e_1}' & \vec{e_2}' \\ | & | \end{bmatrix}$$

Remembering eq. (8), we recognize that we now need to solve for the scalars that multiply each of the new basis vectors, to produce the same physical vector as before:

$$\begin{bmatrix} | & | \\ \vec{e_1}' & \vec{e_2}' \\ | & | \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \equiv \vec{v} \tag{25}$$

Solving, we find that $\vec{v}' = \begin{bmatrix} 2 \\ 14 \end{bmatrix}$. We show this in fig. 16.

Note that in both cases, we're choosing a new basis; the only question is whether we preserve the *coordinates* (which are the mathematical representation) and change the physical vector in accordance with the new basis, or preserve the *physical vector* and find the new coordinates. This can be pretty conceptually confusing, but hopefully figs. 14 to 16 help.

The question now is, how do we solve eq. (25) in general? If given a vector $\vec{v}$ expressed with coordinates in the standard n-dimensional basis $\mathbf{E}$, then we can solve for the coordinates $\vec{v}'$ in a different n-dimensional basis $\mathbf{A}$ with:

$$\vec{v}' = \mathbf{A}^{-1}\vec{v} \tag{26}$$

$\mathbf{A}$ is formed with the columns of the new basis. To do the opposite (basis $\mathbf{A} \rightarrow \mathbf{E}$), we apply $\mathbf{A}$:

$$\vec{v} = \mathbf{A}\vec{v}' \tag{27}$$

Finally, suppose that we're given $\vec{v}$ with coordinates in an arbitrary basis $\mathbf{P}$ (matrix with columns $\vec{p_1}, \ldots, \vec{p_n}$), and we want to change to another arbitrary basis $\mathbf{Q}$ with columns $\vec{q_1}, \ldots, \vec{q_n}$. We must apply the transformation $\mathbf{Q}^{-1}\mathbf{P}$. This follows
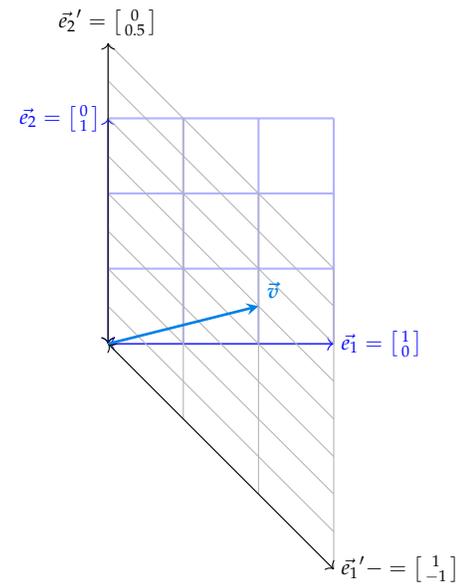
Figure 15: New + Standard Bases (same coordinates): notice that $\vec{v_1}$ has the same coordinates as it did in the Standard Basis (2 steps of $\vec{e_1}'$, 5 steps of $\vec{e_2}'$) but the physical vector is different. To show what the vector looks like in the previous basis, the standard basis grid is included (blue).
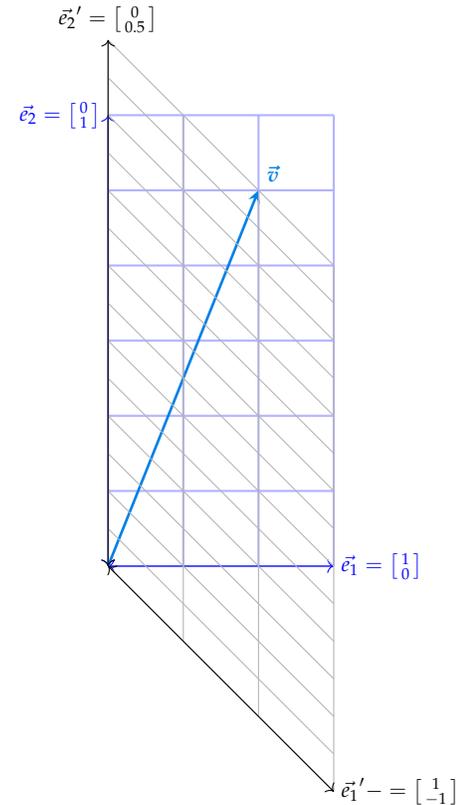
Figure 16: New + Standard Bases (same physical vector): notice that $\vec{v_1}$ is now physically the same as in the Standard Basis but we had to solve for the coordinates in the new basis. The standard basis grid is again in blue.

from first transforming $\vec{v}$ to the standard basis from $\mathbf{P}$, and then transforming to the new basis $\mathbf{Q}$.[67]

[67] Recall that we left-multiply vectors with matrices to apply linear transformations.

Matrix Change of Basis: If you have a good conceptual grasp of the ideas presented above, then this part will be quite intuitive. We can apply our knowledge to linear transformations to understand what it means to change the basis of a matrix. Given transformation $\underset{n \times n}{\mathbf{T}}$ with input $\vec{v_1}$ and output $\vec{v_2}$, we can apply our column-wise interpretation of the matrix-vector product as in eq. (8), but now, we must take care of the fact that our vectors may lie in a different basis than $\mathbf{E}$.

Suppose we have basis vectors $\vec{a}_1, \dots, \vec{a}_n$ forming $\underset{n \times n}{\mathbf{A}}$, and vectors $\vec{v_1}$, $\vec{v_2}$ represented in this basis: $\vec{v_i} = v'_{i,1}\vec{a}_1 + \cdots + v'_{i,n}\vec{a}_n$.[68] We can also represent $\mathbf{T}$ in this basis as $\mathbf{T}'$.

[68] Each $v_{i,k}$ is the kth element of the vector $\vec{v_i}$.

We start with $\vec{v_1} = \mathbf{A}\vec{v_1}'$ and $\vec{v_2} = \mathbf{A}\vec{v_2}'$ as follows from eq. (27). From there, since $\mathbf{T}\vec{v_1} = \vec{v_2}$ by definition of linear transformation $\mathbf{T}$, we can plug in: $\mathbf{TA}\vec{v_1}' = \mathbf{A}\vec{v_2}' \implies \mathbf{A}^{-1}\mathbf{TA}\vec{v_1}' = \vec{v_2}'$. Clearly, just as how we had $\mathbf{T}\vec{v_1} = \vec{v_2}$, we have an analogous transformation in the new basis: $\mathbf{T}'\vec{v_1}' = \vec{v_2}'$, where $\mathbf{T}' = \mathbf{A}^{-1}\mathbf{TA}$. The diagram adapted the course notes in fig. 17 clarifies that $\mathbf{T}'$ is equivalent to $\mathbf{A}$, then $\mathbf{T}$, then $\mathbf{A}^{-1}$. Just follow the arrows!

Diagonalization: What's the point of learning to change bases? The standard basis is perfectly nice and intuitive, so it seems like we're needlessly adding complexity. But in practical applications, matrix operations form the core of some *very* computationally heavy algorithms. This means that we need to be able to easily invert matrices, raise them to high powers, and multiply them commutatively; all of these can be accomplished with *diagonal matrices*, where all entries not on the diagonal are zero.

Figure 17: Diagram showing how to compose various matrices to go from one node to any other. Remember to multiply successive transformations on the left!

First, we must choose a *diagonalizing basis* $\mathbf{A}$ that consists of the n eigenvectors of $\mathbf{T}$. This is only possible if $\mathbf{T}$ is *diagonalizable*, which requires it to have n linearly independent eigenvectors.[69] Then, to figure out how $\mathbf{T}$ transforms $\vec{v_1}$, we write $\vec{v_1} = v'_{11}\vec{a}_1 + \dots + v'_{1n}\vec{a}_n$ and compute:

[69] Note this is different from the *n columns* being independent!

$$\mathbf{T}\vec{v_1} = \mathbf{T}v'_{11}\vec{a}_1 + \cdots + \mathbf{T}v'_{1n}\vec{a}_n$$
$$= v_{11}\left(\mathbf{T}\vec{a}_1\right) + \cdots + v_{1n}\left(\mathbf{T}\vec{a}_n\right)$$
$$= v_{11}\left(\lambda_1\vec{a}_1\right) + \cdots + v_{1n}\left(\lambda_n\vec{a}_n\right)$$

The last step follows from eq. (18), the definition of $(\vec{v}, \lambda)$ pairs. Now, we apply

eq. (8) in a clever way, using the middle matrix (diagonal!) as a sort of "sifting" matrix; each row only has one nonzero value, so we only multiply the $v_{1i}\vec{a}_i$ value by the corresponding λ_i.

$$\mathbf{T}\vec{v_1} = \begin{bmatrix} | & | & & | \\ \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & & & & \vdots \\ \vdots & & & & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_n \end{bmatrix}$$

Notice now we have an equation of the form:

$$\mathbf{T}\vec{v_1} = \mathbf{A}\mathbf{D}\vec{v_1}'$$

$$\mathbf{T}\vec{v_1} = \mathbf{A}\mathbf{D}\mathbf{A}^{-1}\vec{v_1}$$

$$\Downarrow$$

$$\mathbf{T} = \mathbf{A}\mathbf{D}\mathbf{A}^{-1}$$

Here, $\mathbf{D}$ is the diagonal matrix containing the eigenvalues of the transformation! One useful result is that:[70]

$$\mathbf{T}^k = \mathbf{A}\mathbf{D}^k\mathbf{A}^{-1} \tag{28}$$

Since $\mathbf{D}$ is easy to raise to high powers ($\mathbf{D}^k$ contains all diagonal entries raised to the kth power), the computation of numerous transformations becomes much easier!

[70] This follows from:
$$\mathbf{T}^k = \left(\mathbf{A}\mathbf{D}\mathbf{A}^{-1}\right)^k$$
$$\mathbf{T}^k = \underbrace{\left(\mathbf{A}\mathbf{D}\mathbf{A}^{-1}\right)\left(\mathbf{A}\mathbf{D}\mathbf{A}^{-1}\right)\cdots\left(\mathbf{A}\mathbf{D}\mathbf{A}^{-1}\right)}_{k \text{ times}}$$
$$\mathbf{T}^k = \mathbf{A}\mathbf{D}\cdot\mathbf{I_n}\cdot\mathbf{D}\cdot\mathbf{I_n}\cdot\mathbf{D}\ldots\mathbf{I_n}\cdot\mathbf{D}\mathbf{A}^{-1}$$
$$\mathbf{T}^k = \mathbf{A}\mathbf{D}^k\mathbf{A}^{-1}$$

Problems

✠ **Problem:** Given $\vec{v}$ in the standard basis has coordinates as given below, what are the coordinates of $\vec{v}$ in the basis formed by vectors $\vec{a_1}$, $\vec{a_2}$, and $\vec{a_3}$?

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{a_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{a_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{a_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Answer: $\vec{v}' = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$

Quick Solution: We apply eq. (26) and the result follows directly. To show the

steps:

$$\vec{v}' = \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

✠ **Problem:** Is it *true* or *false* that transformation **T** with eigenvectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ as below is diagonalizable?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Answer: True

Quick Solution: We need to check if the eigenvectors are linearly independent. We can perform Gaussian Elimination to determine if they are linearly independent:

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 1 & 2 & 1 & | & 0 \\ 2 & 4 & 0 & | & 0 \end{bmatrix}$$

$$\begin{array}{c} R_2 - R_1 \to R_2 \\ R3 - 2R_1 \to R_3 \end{array} \implies \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 2 & 1 & | & 0 \\ 0 & 4 & 0 & | & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \implies \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 4 & 0 & | & 0 \\ 0 & 2 & 1 & | & 0 \end{bmatrix}$$

$$\begin{array}{c} \frac{1}{4} R_2 \to R_2 \\ R3 - 2R_2 \to R_3 \end{array} \implies \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

We have an identity matrix on the right, so the vectors are linearly independent, and **T** is diagonalizable.

✠ **Problem:** Using diagonalization if possible, compute $\mathbf{T}^k$.

$$\mathbf{T}^k = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}^k$$

Answer: $\mathbf{T}^k = \begin{bmatrix} 1 & 0 \\ 1 - 0.5^k & 0.5^k \end{bmatrix}$

Quick Solution: We follow the diagonalization procedure: first, compute the $(\vec{v}, \lambda)$ pairs of $\mathbf{T}$. Then, ensure the eigenvectors are linearly independent. If they aren't, then $\mathbf{T}$ can't be diagonalized! Compose the matrix containing the eigenvectors as columns ($\mathbf{A}$), which is the diagonalizing basis. Also compute it's inverse. Finally, form $\mathbf{D}$, the diagonal matrix containing the eigenvalues. Raise $\mathbf{D}$ to the kth power, then compute $\mathbf{T}^k = \mathbf{A}\mathbf{D}^k\mathbf{A}^{-1}$. We carry out these steps in the LONG SOLUTION, and the answer should follow: $\mathbf{T}^k = \begin{bmatrix} 1 & 0 \\ 1 - 0.5^k & 0.5^k \end{bmatrix}$

Long Solution: We solve eq. (22): $\lambda^2 - (1 + 0.5)\lambda + (0.5 - 0) = 0 \implies \lambda_1 = 1, \lambda_2 = 0.5$. Plugging into eq. (20), we can solve for each $\vec{v}_i$ of $\mathbf{T}$:[71]

$$\texttt{null}\left(\begin{bmatrix} 0 & 0 \\ 0.5 & -0.5 \end{bmatrix}\right) \implies \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\texttt{null}\left(\begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \end{bmatrix}\right) \implies \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

[71] If these steps aren't clear, I recommend checking out the procedure to compute a null-space.

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent, so we proceed. We can form the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Raising $\mathbf{D}$ to the kth power yields:

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5^k \end{bmatrix}$$

since we can directly raise the entries individually to the kth power for a diagonal matrix. Now, we compute using eq. (28):

$$\mathbf{T}^k = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 - 0.5^k & 0.5^k \end{bmatrix}$$

Practice Set 5: Introduction to Circuit Analysis

Relevant Equations/Information

Circuit Terminology and Elements: There are 3 key quantities to be familiar with before analyzing a circuit *schematic* or *diagram*, such as the one in fig. 18.

- Current [A, amperes]: The flow of charge Q through wires and circuit elements in a certain amount of time.[72] For example, a wire may have 1 of current flowing *through it*. The formula relating current to charge is:

$$I = \frac{dQ}{dt} \tag{29}$$

- Voltage [V, volts $\equiv \frac{J}{C}$]: Potential energy [joule, J] needed to move charge [coulomb, C] from one point to another. A single scalar value called the *potential* is associated with every point in a circuit, and differences in potential are called voltages. We almost always only care about voltages, not potentials. When talking about the voltage *across* a circuit element, we refer to the difference in potentials at the ends of the circuit element.

   A possible point of confusion: when we say (or you read) that a specific point has a certain voltage, this seems to contradict the above, since we don't have two potential values to subtract to get a voltage. But in these cases, the implicit reference point is *ground* ($\perp$), which is at the zero potential point, so the voltage between a point and ground is identical to the potential at that point.

- Resistance [Ω, ohms]: A material's innate chemical structure determines how easily it lets charges move through it. The more restrained the movement, the higher the resistance. This is modeled by resistor R in fig. 18; a more realistic model is in fig. 19. There is a formula for R that accounts for the dimensions of a material, in addition to its innate chemical properties:

$$R = \frac{\rho L}{A} \tag{30}$$

[72] Note it's the negative charges (electrons, e^-) as opposed to the positive charges (protons) that actually move, but the direction of a current arrow in a schematic denotes the direction in which positive current flows.
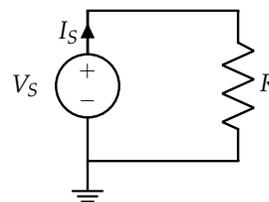
Figure 18: An example circuit to help form an association between terms and symbols. We'll begin analyzing circuits like these soon.
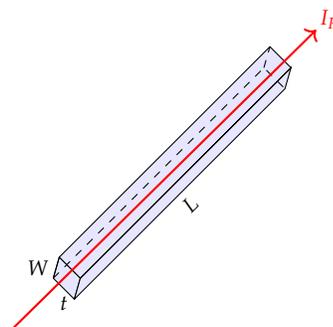
Figure 19: A simple model of a resistor with resistivity ρ, length L, cross-sectional area $W \cdot t$. Note the direction of current flow; the same physical object can have different resistances in different directions (such as if current flowed from top to bottom).

ρ = material's intrinsic resistivity, L = length, A = cross-sectional area. The longer a resistive material, the less easily charges can pass through it. On the other hand, the wider the area, the easier it is to pass through. ρ is typically a constant in this class, though it can sometimes vary with temperature.

What follows is a brief description of each circuit element you'll need to know about; table 1 presents a condensed summary of the relevant features for each and presents this information visually.

1. *Voltage Source*: A voltage source enforces that a specific voltage is always present across its $+$ and $-$ terminals, no matter what the rest of the circuit looks like. The *polarity* (or direction) of the source is given by the $+$ and $-$ signs, and the potential at the $+$ terminal is always V_S greater than at the $-$ terminal. A voltage source can support any current, the value of which is set by the rest of the circuit.

2. *Current Source*: A current source always drives I_S amount of current to flow in the direction of the arrow (as seen in table 1) regardless of the rest of the circuit. The voltage across a current source is determined by the rest of the circuit.

3. *Resistor*: The I-V relationship of a resistor is called Ohm's Law, which states that:

$$V = IR \tag{31}$$

This formula can be solved for voltage or current individually, as in table 1.

4. *Wire*: A wire "drops" zero voltage across it (so the voltage between any two points on the *same wire* is always zero).[73] An ideal wire can support any current.

5. *Open Circuit*: No current is allowed to flow through an open circuit, as indicated by the gap, but the voltage drop across two terminals of an open connection can be anything, as set by the rest of the circuit.

[73] Alternatively, all points on the same wire are at the same potential. This will connect to the idea of nodes and supernodes presented below.

Definition of a Node: A node, sometimes called a junction, is any place in the circuit where two or more of the elements in table 1 meet. Nodes are also, by definition, all at the same potential. Sound familiar? Above, we said that all points on a wire are also at the same potential, so a simple way to identify nodes in a circuit is to trace the portions that are connected by wires. For two points to be part of the same node, they must have a path between them consisting of *only* wire. Also, all wire segments are part of some node, so initially, highlighting portions of a circuit different colors can help. See fig. 20 and note how there are two nodes in the circuit. fig. 21 is a more complex example, but the exact same logic applies.

Name	Symbol	V_{elem}	I_{elem}	I-V Characteristic
Voltage Source		V_S	Any	
Current Source		Any	I_S	
Resistor		$I_{\text{elem}}R$	$\frac{V_{\text{elem}}}{R}$	
Wire		0	Any	
Open Circuit		Any	0	
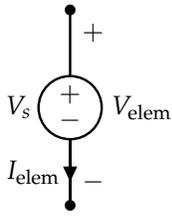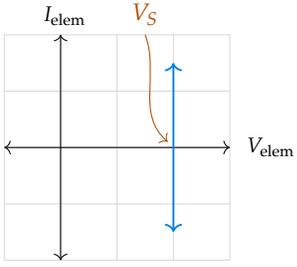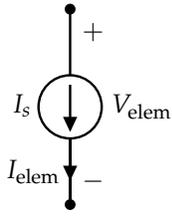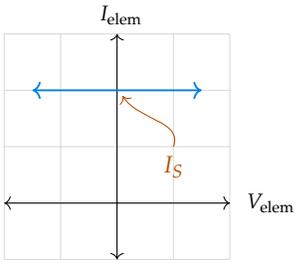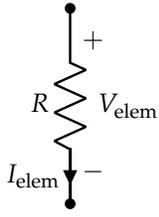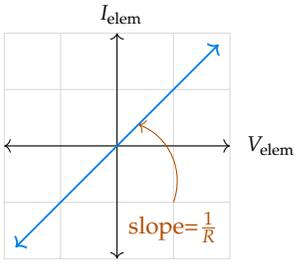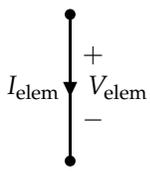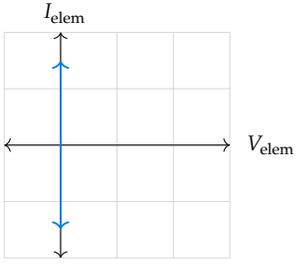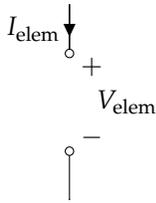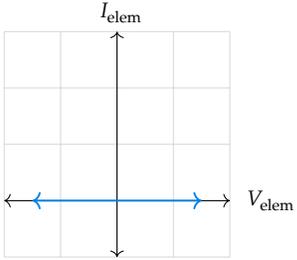
Table 1: A list of all the circuit elements we'll see, including their symbols and IV characteristics. For an explanation of how these graphs were derived, what they mean, and why they match the element's behavior, check the previous page.

Passive Sign Convention (PSC): Take a look at the resistor labeling of current and voltage for fig. 20. In particular, the current I_R comes out of the $-$ terminal of R and would enter the $+$ terminal at the top. This labeling mechanism is very important to keep consistent, and it is called *passive sign convention*. A resistor is a passive element; it just lets charges flow through it, changing their potential energy in the process, so when labeling the voltage and current for this element, we must be self-consistent; if not, our analysis of circuits will be wrong. Notice that the voltage source seems to violate passive sign convention; the current I_S is certainly not coming out of the $-$ terminal of V_S! However, voltage and current sources can be though of as active; they do not follow passive sign convention.

Figure 20: Note that there are two nodes in this circuit, since there are two sections of wire separated by circuit elements. We can get anywhere from blue → blue without crossing circuit elements, but we cannot pass blue → red similarly.

KCL (Kirchoff's Current Law): All currents entering a node/junction must leave the junction. This follows from the physical principle that charge is not accumulating at any point in the circuit over time. Take a look at the magenta node in fig. 21: here, $I_0 = I_1 + I_2$. Mathematically:

$$\sum i_{\text{in}} = \sum i_{\text{out}} \qquad \text{(for each node)} \tag{32}$$

KVL (Kirchoff's Voltage Law): If we choose a closed loop (*any closed loop*) in our circuit and go around it, adding up the voltages as $V_+ - V_-$ (making sure to keep the signs correct), the sum will always be 0V. That is:

$$\sum_{\text{closed loop}} V_i = 0 \tag{33}$$

Let's take a look at an example, using fig. 20. Start at the bottom left corner; we are at a point on the wire, and our total voltage is 0. As we go up and include the voltage source, we go from the $-$ terminal to the $+$ terminal and the magnitude is 5V. In this case, we are going from lower potential to higher potential by crossing the voltage source so our total voltage *increases* by 5V. Then, we cross the top wire; no change in voltage occurs. When we come down across the resistor, the potential *decreases* since we go from higher potential to lower potential. Therefore, our total voltage decreases by V_R.

Figure 21: There are 5 nodes in this larger circuit! This is a variant of the wheatstone bridge, a very common circuit used in sensor-based measuring applications. It's likely to show up if you begin working more with circuits.

We now arrive back at the same blue node we started at (we cross another wire, which contributes zero drop in potential). By KVL in eq. (33), our equation is $5\text{V} + V_R = 0$. Therefore, $V_R = -5$; we lose 5V of potential when crossing the resistor. From a physical standpoint, this makes sense because the charges have less potential

energy at the bottom of the resistor compared to at the top; it took energy to go through R.

As another example (or rather, a few), from fig. 21 and PSC we have: $5V - V_{R_0} - V_{R_2} - V_{R_5} = 0$, $-V_{R_3} - V_{R_4} + V_{R_5} = 0$, and $-V_{R_1} - V_{R_4} + V_{R_5} + V_{R_2} = 0$. Note that these are all closed loops, and we follow PSC.

Systematic Circuit Analysis through Nodal Voltage Analysis: By applying KCL to the nodes in a circuit, and taking advantage of the circuit elements' own formulas (such as Ohm's Law for resistors), we can formulate a system of equations that can be solved for all the useful information in a circuit.[74] This is commonly referred to as *Nodal Voltage Analysis*, or NVA. Using NVA, we can find the potential at all nodes, and the currents through all elements. A thorough worked example carrying out this analysis more completely, from setting up the system of equations to solving it numerically, is described in the next chapter.[75]

Algorithm Steps:

1. *Select a Ground Node*: Arbitrarily select one of the circuit's nodes to assign as a 0V ground. Typically this is the $-$ terminal of the voltage source.

2. *Label the "Easy" Node*: If the $-$ terminal of a voltage source is ground, we can directly label the potential of the $+$ terminal as V_S.

3. *Label other Nodes*: Assign node labels of the form u_i for all unlabeled nodes.

4. *Label Element Voltages and Currents (PSC)*: For all circuit elements, choose an arbitrary current direction.[76] Then, label voltage polarities consistently with PSC.

5. *KCL For Unknown Nodes*: For each node, write out the KCL equation.

6. *Apply Ohm's Law*: For each circuit element except voltage sources (for which the current will be determined by the rest of the circuit), solve for current using node potentials and element values (such as resistance values).

7. *Make Substitutions back into KCL*: Take these current expressions in terms of other variables, and plug back into the node KCL equations.

8. *Form a Matrix and Solve*: Treating every unknown (node potential) as a variable, form an augmented matrix to reduce! Any terms not containing a node potential will be treated as constants.

[74] The standard linear algebraic principles from the previous chapters can be applied to solve this system, including Gaussian Elimination. For example, if you create an augmented matrix, proceed with Gaussian Elimination, and arrive at an inconsistent row, then the corresponding circuit cannot exist, or there is a contradiction somewhere!

[75] Remember this: NVA, applied correctly, can *always solve* a circuit completely. If faced with a problem where simplifications are hard to apply, begin with NVA. In the next chapter, I select an example that's a bit more complicated than a voltage divider; it might be a good idea to review the voltage divider example from the course notes first. The example here is also a good one.

[76] If your choice of current direction for any elements are "wrong," the final answer will simply be negative, so no harm done.

Problems

✠ **Problem:** Is it *true* or *false* that any amount of current can go through a voltage source?

Answer: True

Quick Solution: By definition, an ideal voltage source will maintain a specified potential difference across its terminals, and can support any current. See the I-V relationship in table 1; note that there is no limit on the current.

✠ **Problem:** Is it *true* or *false* that any amount of voltage can be dropped across a current source?

Answer: True

Quick Solution: As above, we apply the definition; an ideal current source will maintain a specified current through itself regardless of the voltage dropped across it. The I-V relationship in table 1 reinforces that there is no limit on the voltage.

✠ **Problem:** Is it *true* or *false* that in fig. 22, V_{R_1} and V_{R_2} are the same?

Answer: True

Quick Solution: We apply KVL going in a closed loop (direction doesn't matter but we choose clockwise) including R_1 and R_2: $V_{R_2} - V_{R_1} = 0 \implies V_{R_1} = V_{R_2}$. Alternatively, if you're like me and prefer forming loops with the voltage source included, we arrive at the same result:

$$\begin{cases} V_S - V_{R_1} = 0 \\ V_S - V_{R_2} = 0 \end{cases}$$

Solving, $V_{R_1} = V_{R_2}$.

Long Solution: More of an extended margin note: If you've studied ahead a bit or are visiting this resource as review, you may recognize that these resistors are *in parallel*. But what exactly does this mean? Any two circuit elements are in parallel if their two terminals share the exact same two nodes.

Here's why it can be useful to recognize when elements are in parallel; by sharing terminal nodes, the elements are forced to have the same voltage drop across them. Consider forming a KVL loop that goes from one node to the other through the first element, but on the reverse path, goes through the other element. The equation is $V_{\text{elem 1}} - V_{\text{elem 2}} = 0$, keeping track of polarities, and the voltage drop must therefore be identical for the two elements. In the same way, we can inductively show that if we can have any number of elements in parallel (sharing end nodes), the voltage drop across all of them must be the same!

✠ **Problem:** Is it *true* or *false* that in fig. 22, if $R_1 = 2\text{k}\Omega$ and $R_2 = 3\text{k}\Omega$, I_{R_1} and I_{R_2} are the same?
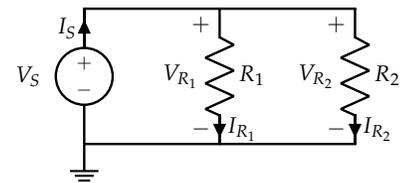
Figure 22: These resistors are connected in *parallel*, meaning their terminals share identical nodes. This is explained more later.

Answer: False

Quick Solution: We apply KVL as in eq. (33) and Ohm's Law as in eq. (31): The previous problem establishes that $V_{R_2} = V_{R_1}$ so we use that result directly and substitute numbers:

$$V_{R_2} - V_{R_1} = 0$$
$$I_{R_2} \cdot 3\text{k}\Omega - I_{R_1} \cdot 2\text{k}\Omega = 0$$
$$I_{R_2} \cdot 3\text{k}\Omega = I_{R_1} \cdot 2\text{k}\Omega$$
$$\frac{I_{R_2}}{I_{R_1}} = \frac{2\text{k}\Omega}{3\text{k}\Omega} \neq 1$$

Evidently, the currents are not equal, so the answer is false. In fact, in the process of answering this question, we have actually derived an important result; see the LONG SOLUTION for more.

Long Solution: A useful principle to keep in mind is that current will divide in such a way that the path of less effective resistance[77] gets proportionally more current. Let's take a simple example: if current I_S enters a junction, and there are two exiting branches of resistances R_1 and R_2 such as in fig. 22, then each branch current is:

$$I_{R_1} = I_S \frac{R_2}{R_2 + R_1} \tag{34}$$

$$I_{R_2} = I_S \frac{R_1}{R_2 + R_1} \tag{35}$$

[77] We'll cover this idea of apparent or equivalent resistance when discussing Thevenin and Norton equivalent circuits.

Notice that if the resistance of R_2 goes up, that branch must receive less of the I_S current (which itself will change). This logic will be made more clear when we begin analyzing circuits in more depth.

✠ **Problem:** In a circuit with n total nodes and k non-wire elements, how many equations (Ohm's Law, KCL, and KVL) do we need to solve the circuit for all node potentials and element currents?

Answer: $n + k - 1$

Quick Solution: We need 1 equation for each variable we want to solve for (follows from all the linear algebra knowledge we have from before). But we also must define a ground node that has a determined potential of 0V. This ground node doesn't *really* constitute an equation, hence the -1 in the answer. For the remaining information, we therefore need $n + k - 1$ equations to solve the system. If you consider setting ground as a reference to be an equation, then we need $n + k$ equations. This is a useful result to remember when performing nodal analysis.

✠ **Problem:** Suppose you're given the circuit in fig. 23. How many nodes do we need to label in order to perform nodal analysis, including ground?

Answer: 4

Quick Solution: To perform nodal analysis, we need to label all nodes in the
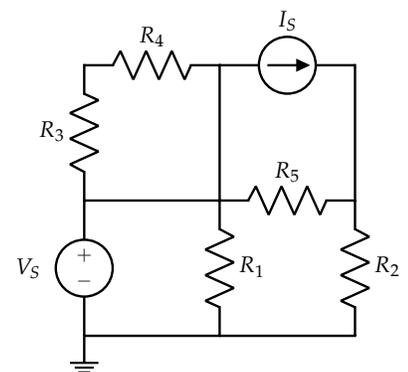
Figure 23: Circuit with 2 independent sources and 5 resistors!

circuit. This is an important result, because otherwise, we won't have enough equations to solve the circuit for a unique solution. Here, there are 4 nodes; they are colored in fig. 24. By labeling these 4 nodes, including the ground node (red), we have a total of 11 equations to work with; 4 from the nodes (we get the potential at ground directly, $u_g = 0$), and 7 from the circuit elements. This would allow us to solve the system!

Figure 24: The 4 nodes are colored!

✠ **Problem:** How many nodes are in the circuit given in fig. 25? The node potentials and currents are for the next problem and don't affect the answer to this question.

Answer: 3

Quick Solution: We follow the same procedure as before; how many sets of wires are there, that are separated by circuit elements? There's the top wire, the ground wire, and the wire between R_1 and R_2.

✠ **Problem:** Suppose you have a labeled circuit diagram, as in fig. 25. Using passive sign convention, add voltage polarities ($+$ and $-$) for R_1 and R_2.

Answer:

Figure 25: Partially labeled voltage divider circuit.

Quick Solution: According to PSC, for passive circuit elements such as resistors, current will flow from the positive terminal to the negative terminal.

Long Solution: The current must be coming out of the negative terminal of each resistor and it will enter the positive terminal. Only if PSC is applied correctly will the current and voltage labeling be self-consistent. For R_1, the current is given to be going from bottom to top, u_2 to u_1, so the top side of R_1 must be negative. Similarly, for R_2, since the current goes top to bottom, the bottom must be the negative terminal.

✠ **Problem:** For the same circuit as above, fig. 25, what are all the equations we can write out to solve the circuit for all node potentials and branch[78] currents?

Answer:

$$I_{R_1} = I_{V_S}$$
$$0 = I_{R_2} + I_{R_1}$$
$$u_1 = V_S$$
$$u_2 - u_1 = I_{R_1} R_1$$
$$u_2 - 0 = I_{R_2} R_2$$

Quick Solution: As per step 2 of the NVA process ("easy" node), we immediately see that $u_1 = V_S$. Then, for each node and circuit element, we can write out the KCL and Ohm's Law Equations, as per the steps to perform NVA. Then, by noticing linear dependence, we arrive at the equations above. See the LONG SOLUTION for details!

Long Solution: We have $u_1 = V_S$. There is now only 1 node potential we do not know; u_2. To be thorough, we can apply KCL to each of the 3 nodes, in the same form as eq. (32):

1. $u_1 : I_{R_1} = I_{V_S}$

2. $u_2 : 0 = I_{R_2} + I_{R_1}$

3. ground : $I_{V_S} + I_{R_2} = 0$

We then write out the resistor equations using Ohm's Law, eq. (31):

1. $R_1 : V_{R_1} = I_{R_1} R_1 = u_2 - u_1$

2. $R_2 : V_{R_2} = I_{R_2} R_2 = u_2 - 0$

Note that we have 6 total equations, but only 5 in the answer. Also, since we have 1 proper (non-ground) node and 3 circuit elements, we should need only 4 equations in addition to the to solve the circuit. This problem shows that we can easily create redundant equations; note that our KCL equation for ground is linearly dependent with the other two KCL equations. After some rearranging, we arrive at the same equations in the ANSWER.

Practice Set 6: Circuit Analysis, Resistive Modeling and Power

Relevant Equations/Information

A Worked NVA Example: Let's systematically analyze the circuit in fig. 26 to solve for all currents and node potentials. I'll go through the steps both symbolically and numerically, using the following component values: $V_S = 12\text{V}$, $R_0 = 2\text{k}\Omega$, $R_1 = 2\text{k}\Omega$, $R_2 = 5\text{k}\Omega$, $R_3 = 12\text{k}\Omega$, $R_4 = 6\text{k}\Omega$, $R_5 = 18\text{k}\Omega$.

1. *Select a Ground Node*: I chose the $-$ terminal of the voltage source here to assign as a reference (see fig. 27), but you could choose *any* node; your results will be self-consistent.

2. *Label the "Easy" Node*: I've colored all four nodes for convenience. Now, the red node at the top has known potential of $u_1 = V_S = 12\text{V}$.

3. *Label other Nodes*: Done!

4. *Label Element Voltages and Currents (PSC)*: These have been added in fig. 28.

5. *KCL For Unknown Nodes*: Alright, now the work begins. Let's write out our 2 KCL equations using eq. (32):

$$u_2 : I_{R_1} = I_{R_2} + I_{R_5}$$

$$u_3 : I_{R_2} = I_{R_3} + I_{R_4}$$

Notice that we omit the ground node KCL equation because it is linearly dependent with the 2 above, and the potential at u_1 is a known quantity.

6. *Apply Ohm's Law*: We have the following circuit element equations:

$$R_0 : V_{R_0} \equiv u_1 = I_{R_0} R_0$$

$$R_1 : V_{R_1} \equiv u_1 - u_2 = I_{R_1} R_1$$

$$R_2 : V_{R_2} \equiv u_2 - u_3 = I_{R_2} R_2$$

$$R_3 : V_{R_3} \equiv u_3 = I_{R_3} R_3$$

$$R_4 : V_{R_4} \equiv u_3 = I_{R_4} R_4$$

$$R_5 : V_{R_5} \equiv u_2 = I_{R_5} R_5$$
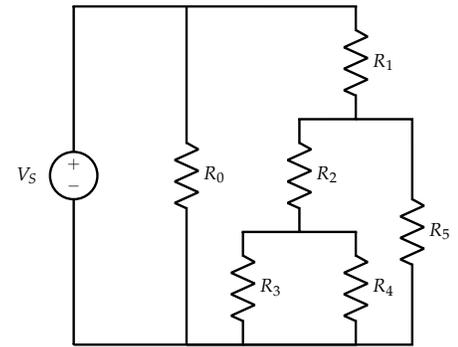
Figure 26: Looks complicated! But we can systematically analyze this circuit just like any another.
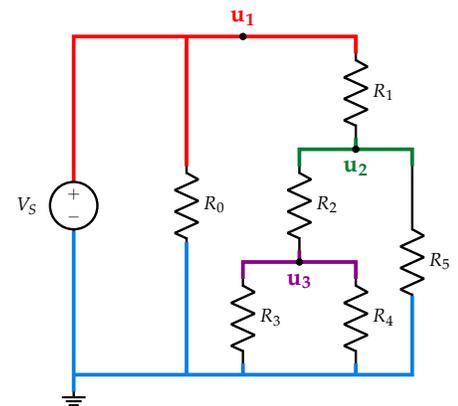
Figure 27: Same as fig. 26 but with colored, labeled nodes.
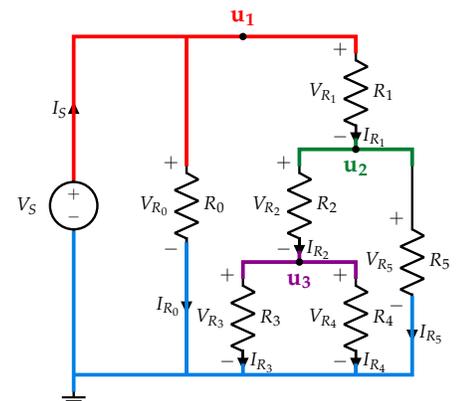
Figure 28: Now with currents and voltages. And yes, it is cluttered in these margins :(

Note that for each voltage, we expressed it in terms of our unknowns, which are the individual node potentials u_2 and u_3.

7. *Make Substitutions back into KCL*: By taking the equations above (dividing both sides by resistance to isolate the current terms) and making substitutions:[79]

$$u_2: \frac{u_1 - u_2}{R_1} = \frac{u_2 - u_3}{R_2} + \frac{u_2}{R_5}$$

$$u_3: \frac{u_2 - u_3}{R_2} = \frac{u_3}{R_3} + \frac{u_3}{R_4}$$

8. *Form a Matrix and Solve*: Now, we have a system of 2 unknowns (the node potentials of u_2 and u_3) and 2 equations! The resistances and the potential at u_1 are known. Let's rearrange terms so the augmented matrix structure is easier to compose:

$$\frac{u_2}{R_2} - \frac{u_3}{R_2} + \frac{u_2}{R_5} + \frac{u_2}{R_1} = \frac{u_1}{R_1}$$

$$\frac{u_2}{R_2} - \frac{u_3}{R_2} - \frac{u_3}{R_3} - \frac{u_3}{R_4} = 0$$

Now, grouping like terms:

$$u_2\left(\frac{1}{R_2} + \frac{1}{R_5} + \frac{1}{R_1}\right) + u_3\left(-\frac{1}{R_2}\right) = \frac{u_1}{R_1}$$

$$u_2\left(\frac{1}{R_2}\right) + u_3\left(-\frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4}\right) = 0$$

The augmented matrix is now:

$$\left[\begin{array}{cc|c} \frac{1}{R_2} + \frac{1}{R_5} + \frac{1}{R_1} & -\frac{1}{R_2} & \frac{u_1}{R_1} \\ \frac{1}{R_2} & -\frac{1}{R_2} - \frac{1}{R_3} - \frac{1}{R_4} & 0 \end{array}\right] \equiv \left[\begin{array}{cc|c} \frac{1}{5} + \frac{1}{18} + \frac{1}{2} & -\frac{1}{5} & \frac{12}{2} \\ \frac{1}{5} & -\frac{1}{5} - \frac{1}{12} - \frac{1}{6} & 0 \end{array}\right]$$

Note that multiplying both equations by 1000 makes the numbers a bit easier to work with. Solving by using Gaussian Elimination, $u_2 = 9V$ and $u_3 = 4V$. Using these and previous information, we can now calculate all the element currents: $I_{R_0} = \frac{12V}{2k\Omega} = 6mA$, $I_{R_1} = \frac{12V-9V}{2k\Omega} = 1.5mA$, $I_{R_2} = \frac{9V-4V}{5k\Omega} = 1mA$, $I_{R_3} = \frac{4V}{12k\Omega} = \frac{1}{3}mA$, $I_{R_4} = \frac{4V}{6k\Omega} = \frac{2}{3}mA$, $I_{R_5} = \frac{9V}{18k\Omega} = 0.5mA$. And we're done!

Voltage Divider Circuit: As it turns out, the voltage divider circuit topology, introduced briefly (and only by name) in the previous chapter, is quite critical to understand. The course notes also analyze the voltage divider circuit (fig. 29) in detail, but the most important thing to recognize is the following:

$$V_{mid} = V_S \frac{R_2}{R_1 + R_2} \tag{36}$$

This is because the current in the circuit is the same everywhere (we have a single loop), and by KVL, the sum of the voltage drops must be 0. That is:
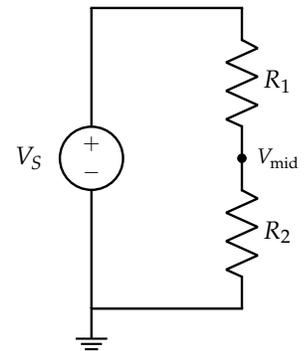
Figure 29: The ubiquitous voltage divider circuit. Get used to seeing and working with this!

$V_S - I_{R_1}R_1 - I_{R_2}R_2 = 0$. Each current can be re-expressed in terms of node potentials: $I_{R_1} = \frac{V_S - V_{\text{mid}}}{R_1}$ and $I_{R_2} = \frac{V_{\text{mid}}}{R_2}$. Solving and rearranging, we arrive at eq. (36).

In this way, we can actually tune our resistor *ratios*[80] to output only a portion of the input, or source, voltage V_S. Note that V_{mid} will always be less than V_S (we would need $R_2 = \infty\,\Omega$ to achieve $V_{\text{mid}} = V_S$). Sometimes, V_{mid} will be written as V_{out} or even V_{o}, since we often measure the voltage dropped across R_2 as our circuit *output voltage*.

1D Resistive Touchscreen: A Whirlwind Summary:[81] The resistive touchscreen is a very practical and powerful application of the concepts covered thus far, especially the voltage divider! Touchscreens are on almost every device, and using the concepts we know, we can start to understand how they work. Often, given a touchscreen, we want to find the position where it is touched. Let's start with a 1D example, as in fig. 30. Finding the touch position on the top (magenta) layer is the same as finding $\frac{L_{\text{touch}}}{L} \equiv \frac{L_{\text{touch}}}{L_{\text{touch}} + L_{\text{rest}}}$.

[81] Please note that this section is a very compressed version of the 1D touchscreen presented in the course notes.

We can refer to eq. (30) in order to approach the touchscreen from a more geometrical perspective. The entire length of the touchscreen can be thought of as a resistor, since the top layer's material has some resistivity ρ. It also has length L_{total}, and let's say that from a side-view, it has area A. Note that this is not the area of the top surface of the touchscreen that the finger is coming from; the relevant area is that where current flows across, and here, the current flows left-to-right across the top conductive layer. We can convert directly from length L to resistance R, since for a given screen, we will have a constant ρ (same material) and constant A. Based on the formula, when the touchscreen is pressed at some point, we can divide the top layer into 2 resistors, each of which is a portion of the whole. This is pictured in fig. 31.

Figure 30: Resistive touchscreen, 1D preliminary model. Notice the connection between the top and bottom conductive layers at the point where the finger touches the screen. Excuse the very poorly tikz-drawn hand, LaTeX should not be used as a substitute for a drawing tool. But at least it has 5 fingers!

We can actually also attach a voltage source to the two ends of the touchscreen and measure the output voltage as the voltage dropped across L_{rest} (see fig. 32). Then, by applying our voltage divider formula, we can develop a relationship between V_{mid} and L_{touch}! Starting with eq. (36), we substitute directly and make simplifications:
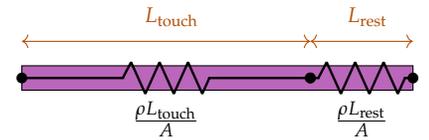
Figure 31: Resistive touchscreen with subresistors explicitly shown.

$$V_{\text{mid}} = V_S \frac{R_{\text{rest}}}{R_{\text{rest}} + R_{\text{touch}}}$$

$$V_{\text{mid}} = V_S \frac{\frac{\rho L_{\text{rest}}}{A}}{\frac{\rho L_{\text{rest}}}{A} + \frac{\rho L_{\text{touch}}}{A}}$$

$$V_{\text{mid}} = V_S \frac{L_{\text{rest}}}{L_{\text{rest}} + L_{\text{touch}}}$$

$$V_{\text{mid}} = V_S \frac{L_{\text{rest}}}{L_{\text{total}}}$$

Note some important results; the 1D touchscreen is very closely connected to the resistive voltage divider; there's no complexity that arises based on the material geometry since all those factors cancel out. Again, only the *ratio* of resistances matters, so the touchscreen can be as big or small as needed for a specific application; the result still holds.

Now, we address the purpose of the bottom black conductive layer; why is it needed? Well, in practice, we will use it to measure V_{mid} by only collecting data from the edge of the screen. If we only had one layer, then to measure V_{mid}, we would need to probe the actual location where the finger contacts the screen! Of course, this isn't feasible. For a more thorough discussion on why adding the bottom conductive layer to the model doesn't affect our previous results, see the co urse notes. The general intuition is that there is no physical connection between the bottom layer and the circuit, so while it enables us to measure certain voltage drops, it doesn't add new paths for current to flow.

Measurement Tools and Techniques: There are two main quantities we may want to measure in a circuit; current and voltage. We have two tools to help us: the ammeter and voltmeter. Given fig. 29, suppose we want to measure the current through R_2 and also the voltage across R_2. We would connect an ammeter as in fig. 33 and a voltmeter as in fig. 34.

Notice how the ammeter blends into the wire after R_2, allowing it to measure the current flowing through itself.[82] A good ammeter must have almost no resistance (resembling a short, or wire); otherwise, its presence will affect the circuit! Measurement tools should remain as close to invisible as possible. If the very act of measuring a quantity changes that quantity, the information we get isn't very useful.

The voltmeter is attached in parallel with the resistor (they share the same terminal nodes).[83] In this case, our voltmeter will need to have a *very* high resistance (so it resembles an open circuit); otherwise, it might "steal" some of the current going through R_2, and the circuit more broadly will be affected.

Power: Power refers to the rate of energy change: $P = \frac{dE}{dt}$ and has units of watts $[W = \frac{J}{s}]$. A single element can *generate* ($P < 0$) or *dissipate* ($P > 0$) energy. We can derive the formula for power in terms of quantities we can measure; current and
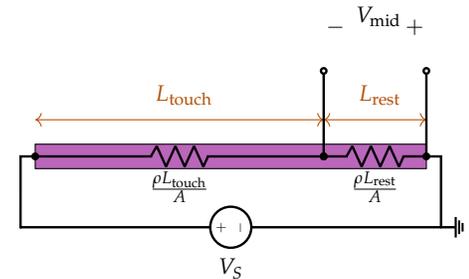
Figure 32: Resistive touchscreen with sub-resistors explicitly shown, along with a voltage source to create a potential difference.
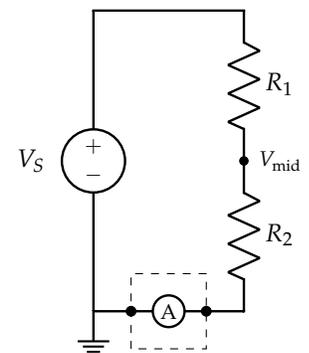
Figure 33: Notice the ammeter (boxed) connected in *series* with R_2.
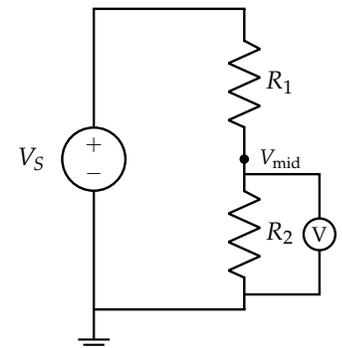
Figure 34: Notice the voltmeter connected in *parallel* with R_2.

[82] By applying KCL at the node between the ammeter and R_2, the current through R_2 must be the same as that going through the ammeter.

[83] We form a small KVL loop involving R_2 and the voltmeter to see that the voltage drop across the resistor will be measured by the voltmeter.

voltage! We can apply some definitions: voltage is potential energy per unit charge, so we can write $V = \frac{dE}{dQ} \implies dE = VdQ$. Differentiating both sides with respect to time yields power on the left: $\frac{dE}{dt} \equiv P = V\frac{dQ}{dt}$. Fortunately, we know that $\frac{dQ}{dt} = I$, so we find that in the steady-state (when current flow itself is constant over time):

$$P = IV \tag{37}$$

Using Ohm's law (eq. (31)) for resistors,[84] we can write eq. (37) in the following equivalent forms:

$$P = IV \equiv \frac{V^2}{R} \equiv I^2 R \tag{38}$$

Now, we address a point of common confusion related to terminology by explicitly outlining the meaning of 4 commonly encountered phrases:

1. negative power dissipated $\implies P < 0$.

2. positive power dissipated $\implies P > 0$.

3. negative power generated $\implies P > 0$.

4. positive power generated $\implies P < 0$.

Problems

✠ **Problem:** Given fig. 35, how many nodes do we need to label to perform NVA (including ground and V_S)?

Answer: 5

Quick Solution: This is very similar to a problem in the previous chapter; we need to label all nodes in the given circuit to perform NVA, of which there are 5. If we don't label all nodes, we will not have enough information (equations) to solve the circuit completely. See fig. 36 for the nodes!

✠ **Problem:** Given a resistive slab such as in fig. 19, by what factor does the resistance change if all dimensions are doubled?

Answer: $\frac{1}{2}$

Quick Solution: Suppose we have $R_0 = \frac{\rho L}{A} = \frac{\rho L}{Wt}$. Then doubling all dimensions L, W, t: $R' = \frac{\rho \cdot 2L}{A'} = \frac{\rho \cdot 2L}{2W \cdot 2t} = \frac{1}{2}\frac{\rho L}{Wt} = \frac{1}{2}R_0$. Therefore, our resistance falls by a factor of $\frac{1}{2}$. Notice that even though the length increased, the cross-sectional area depends on 2 of the dimensions each of which doubled, so the increase in area has a greater impact on the resistance of the slab.

[84] One important result to remember is that by passive sign convention, resistors always dissipate power because current enters the $+$ terminal. Voltage sources tend to generate power, since current comes *out* of the $+$ terminal (and the product $P = IV < 0$).
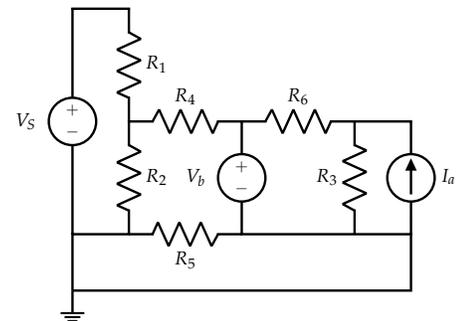
Figure 35: Circuit for the first problem!
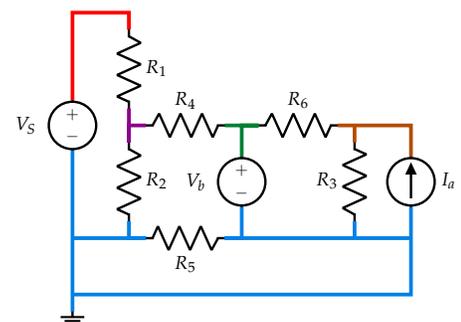
Figure 36: And now with colored nodes!

✠ **Problem:** Given a cylindrical piece of metal with length $l = 10$cm and radius 1mm, you take a resistance measurement across the length, between the circular ends, and find $R_{\text{cyl}} = 0.54$mΩ. What is ρ_{metal}?

Answer: $1.7 \cdot 10^{-8} \Omega\,\text{m}$

Quick Solution: Starting from eq. (30), we rearrange to isolate ρ: $R = \frac{\rho L}{A} \implies \rho = \frac{RA}{L}$. Now, we no longer have a rectangular prism-shaped slab; our cylinder has a different cross-sectional area. Calculating (making sure our units are correct!): $A = \pi r^2 = \pi \left(10^{-3}\text{m}\right)^2 = 3.14 \cdot 10^{-6}\text{m}^2$. Plugging in the remaining two given values:

$$\rho = \frac{\left(0.54 \cdot 10^{-3}\Omega\right) \cdot \left(3.14 \cdot 10^{-6}\text{m}^2\right)}{0.1\text{m}} = 1.69 \cdot 10^{-8} \Omega\,\text{m} \approx 1.7 \cdot 10^{-8} \Omega\,\text{m}$$

✠ **Problem:** Given fig. 37, what is the voltage drop across R_1?

Answer: 0V

Quick Solution: Notice that R_1 is in parallel with a wire (short); since no voltage can be dropped across any ideal wires, and elements in parallel have the same voltage drop, the drop across R_1 must be zero.

Long Solution: There are at least 2 other ways to intuitively see the same result as the QUICK SOLUTION. As we have done before, we can form a KVL loop including R_1 as the only non-wire circuit element, such that the voltage drop across R_1 is zero by applying eq. (33).

More intuitively, when current leaves the voltage source and reaches the junction above R_1, it has two possible exit branches; 1) the wire and 2) the resistor. Current will always prefer the path of less resistance and here, the resistor has infinitely more resistance compared to the wire! So none of the incoming current will go through R_1, and by Ohm's law, if there is no current ($I_{R_1} = 0$), then there is no voltage drop ($V_{R_1} = 0$). We reach the same result by applying the current divider equation (eq. (34)): $I_{R_1} = I_S \frac{R_{\text{wire}}}{R_{\text{wire}} + R_1} = 0$.

✠ **Problem:** Which of the resistors in fig. 38 have voltage drop V_S across them?

Answer: R_1, R_2, R_3, R_5 (all but R_4)

Quick Solution: This question amounts to noticing which resistors can be part of carefully chosen closed KVL loops where the only other circuit element is V_S. Using the node labeling, we list these loops:

$$R_1 : a \to b \to c \to f \to a$$
$$R_2 : a \to b \to c \to d \to e \to g \to f \to a$$
$$R_3 : a \to b \to c \to d \to g \to f \to a$$
$$R_5 : a \to b \to c \to g \to f \to a$$
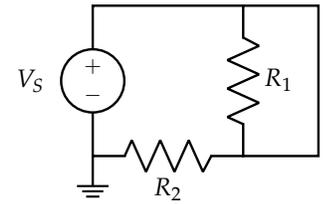
Figure 37: Diagram for the problem to the left.
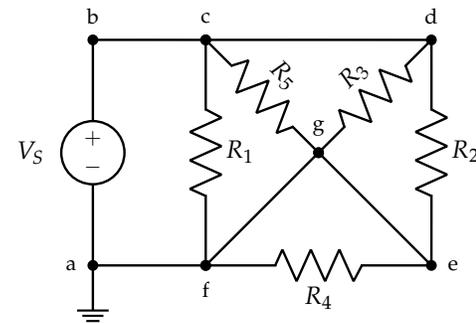
Figure 38: Diagram for the problem to the left.

For R_4, we cannot arrive at such a loop. We can show this by noting that the two terminals of R_4 are e and f. The only paths to reach e from the node above V_S are $c \to g \to e, c \to d \to e, c \to f \to g \to e$, and $c \to g \to d \to e$. These all contain at least 1 resistor and so no closed loop can be formed that contains solely R_4 and V_S.

Long Solution: Full NVA for a circuit like this might be tedious, but will yield the correct answer. The especially diligent student who takes this approach (or one more experienced in inspecting circuits) will find the following node potentials [V]:

$$a, f, g, e : 0$$
$$b, c, d : V_S$$

The element currents are (using the given $+, -$ terminals and node potentials above):

$$R_1(+ : c, - : f) \implies I_{R_1} = \frac{V_S}{R_1} \text{A}$$
$$R_2(+ : d, - : e) \implies : I_{R_2} = \frac{V_S}{R_2} \text{A}$$
$$R_3(+ : d, - : g) \implies : I_{R_3} = \frac{V_S}{R_3} \text{A}$$
$$R_4(+ : e, - : f) \implies : I_{R_4} = 0 \text{A}$$
$$R_5(+ : c, - : g) \implies : I_{R_5} = \frac{V_S}{R_5} \text{A}$$

Notice also that R_4 is the only resistor with both terminals in the same "group" of node potentials, so the difference in potential across both ends is zero, not V_S as for the others.

✠ **Problem:** Suppose we modify the circuit above to add R_6 between g and f as shown in fig. 39. Now, which of the resistors have voltage drop V_S across them?

Answer: Only R_1

Quick Solution: Going through the exact same logic as before, we find that the KVL loop for R_1 remains the same, but all other resistor loops relied on the $g \to f$ connecting being a short in order to contain only the voltage source and a single resistor. Adding R_6 makes it so each of these others loops now contains an additional resistor, so we cannot simply assert that those voltage drops are still zero.

Figure 39: Diagram for the problem to the left.

✠ **Problem:** Suppose you're told a particular circuit element's power is negative; what does this say about the element if it is a resistor? What if it's a voltage source? How about if it's a current source? The options in each case are: *nonsensical* (cannot be true), power is *generated*, and power is *dissipated*.

Answer: Resistor: *nonsensical*, Voltage/Current Source: power *generated*

Quick Solution: By definition of passive-sign convention, the power for a resistor cannot be negative; that is, resistors *always* dissipate power as explained in the RELEVANT INFORMATION section. However, current sources and voltage sources are not passive elements and do not follow PSC. Here, since $P < 0$, the sources are generating power.[85]

✠ **Problem:** Given fig. 40, what is the power consumption of R_2?

Answer: 2W dissipated

Quick Solution: Applying the voltage divider formula eq. (36), $V_{R_2} = 6 \cdot \frac{2}{4+2} = 2V$. For a resistor, out of the 3 equivalent forms in eq. (38), a convenient formula to apply here is $P_{R_2} = \frac{V_{R_2}^2}{R_2} = \frac{4}{2} = 2W$. So 2 watts are dissipated by R_2.

It might be simpler to note that using NVA (or visual inspection), the current in the circuit (same everywhere) is $I = 1A$. So $P_{R_2} = I_{R_2}^2 R_2 = 1A \cdot 2\Omega = 2W$.

✠ **Problem:** For the same circuit as before, fig. 40, what is the power consumption of $V_S = 6V$?

Answer: 6W generated $\equiv -6W$ dissipated

Quick Solution: After solving for the circuit current as $I = 1A$, as before, we note that the voltage source consumes $P = IV = 1A \cdot (-6V) = -6W$ so it actually generates 6W.[86]

Figure 40: Notice the topology of this circuit?

[85] In reality, voltage/current sources are actually quite complex elements in physical systems, and they have to be to reliably enforce certain features that a circuit designer can depend on.

[86] Yet another solution to the previous problem involves proportions, now that we've calculated P_{V_S}; of the generated 6W, R_2 dissipates half as much as R_1 and the total dissipation for the circuit must be 0W since energy must be conserved over time. Therefore, R_2 dissipates $\frac{1}{3}(6W) = 2W$.

Practice Set 7: Thevenin and Norton Equivalence

Relevant Equations/Information

2D Touchscreen: For this part, I highly recommend the course notes; much of the intuition from the 1D touchscreen carries over, and the extension into another dimension requires solid visuals, which the course notes have, and would take me too long to generate. Also, the practice sets don't have any specific questions on the 2D touchscreen; recall that the touchscreen is an interesting application of fundamental concepts we've already covered!

Dependent Sources: As briefly mentioned before, the voltage and current sources covered so far have been *independent* sources, meaning they have a constant, fixed value. However, *dependent sources* also generate currents and voltages but their output depends on some other "controlling" current i_c or voltage v_c in the circuit. There are 4 varieties of dependent sources, as listed in figs. 41 to 44.

Superposition: When there are multiple independent sources in a circuit, such as fig. 35 or fig. 23, we can simplify circuit analysis using the principle of *superposition*. Simply stated, we can consider only one independent source at a time by "zeroing out" all others and analyzing the circuit. We will arrive at some node potentials and some element currents associated with that source (and that source alone). Then, we repeat for the other sources. At the very end, the principle of superposition says that we can simply add all the individual values to arrive at the overall circuit's potentials and currents!

What does it mean to "zero out" a source? For a voltage source, this means no voltage is dropped, and observing table 1, we note this behavior is just like a wire. For a current source, no current can flow through it, just like an open circuit. Therefore, we can use wires or open circuits to zero out sources depending on whether they're supplying voltage or current!

To understand why superposition is even valid, we refer to our properties of
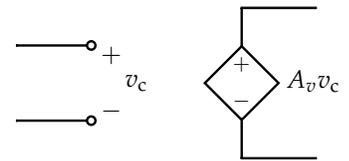
Figure 41: Voltage Controlled Voltage Source (VCVS) (unitless voltage gain A_v)
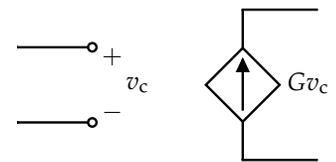
Figure 42: Voltage Controlled Current Source (VCCS) (transconductor with transconductance G (units: Siemens, S, or $\frac{1}{\Omega}$)
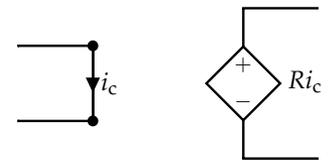
Figure 43: Current Controlled Voltage Source (CCVS) (transresistor with resistance R)
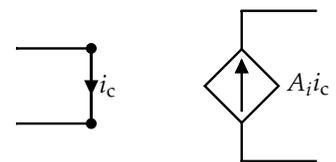
Figure 44: Current Controlled Current Source (CCCS) (unitless current gain A_i)

linear algebra. Note that the NVA analysis procedure culminates in solving an augmented matrix, such that each unknown node potential we solve for can be expressed as a linear combination (the scalars come from the resistor values) of the input variables (independent sources). This follows directly from eq. (8), left-multiplying both sides by $\mathbf{A}^{-1}$ to isolate the vector of unknowns, $\vec{x}$. This can be done no matter how many independent sources we have. The principle of superposition says that we can explicitly decompose a circuit into the contributions of each source on each unknown and add them back again linearly at the end to get the result.

Equivalent Circuits: In many cases, a complex combination of independent sources and resistors can be simplified into an *equivalent circuit* which has the same I-V characteristics as the original but is easier to work with.[87] So far, we have only seen circuits and circuit elements that have linear I-V relationships, and these lines are defined by their x- and y-intercepts. In context, these are:

1. Open-circuit voltage: voltage $V_{\text{open}} \equiv V_{\text{th}}$ when $I = 0$. Used for thevenin equivalent, as in fig. 45.

2. Short-circuit current: current $I_{\text{sc}} \equiv I_N$ when $V = 0$. Used for norton equivalent, as in fig. 46.

Once we find these two special values for any circuit, we can create its I-V relationship and use that to construct one of 2 kinds of equivalent circuits. Note that an equivalent circuit exists between 2 terminals; see fig. 47. Suppose our circuit has an I-V relationship as given in fig. 48, when looking between the 2 output terminals. Let's take a look at how to actually find the specific values:

Thevenin Equivalent Circuit: To find V_{th}, we recognize that it equals the difference in node potentials at a and b, so performing NVA on the original circuit will yield V_{th}. As for finding R_{th}, see below.

Norton Equivalent Circuit: To find I_N, we short terminals a and b and measure the resulting current. We describe how to solve for R_N below.

Note that the thevenin and norton equivalents are not unique for a given circuit; the same circuit can look quite different from different output terminals. For examples deriving the thevenin and norton equivalent for various sample circuits, see the course notes.

Procedure to find $R_{eq} \equiv R_{th} \equiv R_n$: There are 3 methods that can be used to solve
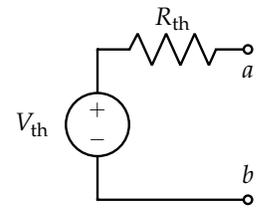
Figure 45: General form of a Thevenin equivalent circuit. Given a circuit and two output terminals, we know the above gives a voltage-source based equivalent; the work lies in solving for V_{th} and R_{th}.
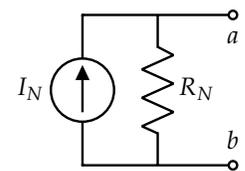
Figure 46: General form of a Norton equivalent circuit, the current-source based equivalent form for a given circuit. Here, we must solve for I_N and R_N.

[87] Note that the I-V characteristic is the *only* feature preserved by an equivalent circuit; extending the equivalence to a concept such as power will lead to mistakes.
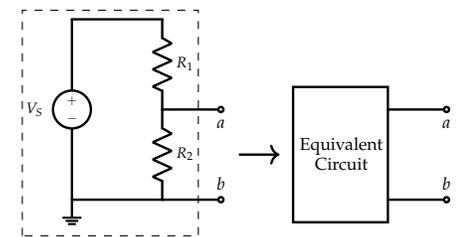
Figure 47: We can form an equivalent circuit for the voltage divider when looking "into" it from terminals a and b.
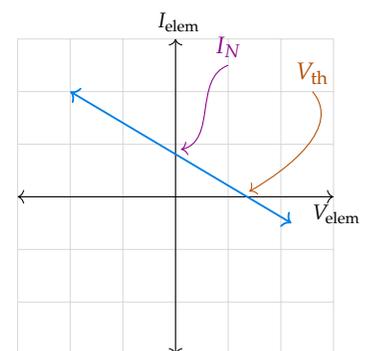
Figure 48: Sample I-V Relationship for a mystery (linear) circuit; we'll use this to construct thevenin and norton equivalents!

for the equivalent resistance between 2 terminals, outlined below:

1. *Apply Test Source*: In order to solve for the effective resistance between the two terminals, we first zero out all *independent* sources. Then, we can "feed in" a test current I_t and measure the corresponding voltage drop across the terminals V_t, or establish a voltage V_t and measure the resulting I_t. Either technique will work; see fig. 49 and fig. 50. The ratio $\frac{V_t}{I_t}$ will yield R_{eq}. See course notes for applications of this technique to simple circuits; we tackle more complex examples in the PROBLEMS. This $\frac{V_t}{I_t}$ technique always works, for any circuit, just like NVA!

2. *Resistor Network Simplifications*: If the resistors between a and b resemble series or parallel configurations, and there are *no dependent sources in the circuit*, the techniques of the section below can be useful. Otherwise, we must resort to test source applications and measurements.

3. *Use V_{th}, I_N*: Suppose we have already derived the equivalent thevenin voltage and norton current for a circuit; then the ratio of these, $\frac{V_{th}}{I_N}$ gives R_{eq} directly! We are effectively taking the slope of the I-V relationship, as in fig. 48. Note that this technique will work *only if there is ≥ 1 independent source* in the circuit; otherwise, we end up with only a point at the origin and cannot solve for R_{eq}.

Note an important consequence of the above; the equivalent resistance for both the Thevenin and Norton variants is always the same! Intuitively, if the equivalent circuits have the same I-V characteristics as the original circuit, and the slope of the line gives $\frac{1}{R_{eq}}$, then the resistance must be the same for all equivalent circuits.

Deriving Parallel and Series Resistance Formulas: Many times, we will see resistors connected together in a complicated way between 2 terminals of interest. But oftentimes, these resistors can be simplified into a less complex network of *equivalent resistors* based on certain rules. We consider two especially useful configurations: Series and Parallel.[88]

Series Resistors: Resistors are said to be in series if by KCL, they have the same current through them. Consider fig. 29 and fig. 37; resistors R_1 and R_2 are in series. In fig. 23, R_3 and R_4 are in series. For any k resistors in series, we can replace them with a single equivalent resistor:

$$R_{eq} = R_1 + R_2 + \ldots + R_k \tag{39}$$

The geometrical intuition for why the resistances add comes from the resistance formula, eq. (30). Resistors in series are essentially components of one very long
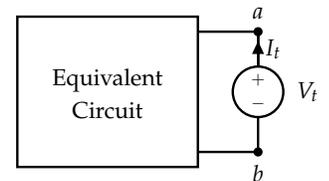
Figure 49: By applying a test voltage V_t across a and b, we can measure the resulting current draw of the equivalent circuit I_t and use that to calculate R_{eq}.
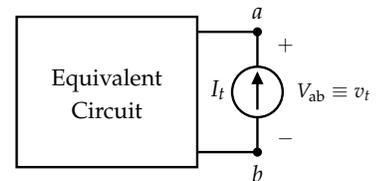
Figure 50: By feeding a test current I_x into the equivalent circuit, we can measure the resulting voltage drop V_{ab} and calculate R_{eq}.

[88] The course notes derive the same information in a more rigorous way with equivalent circuits; I try to provide a more geometrical intuition below.

resistor! Adding many resistors in series *always increases* the equivalent resistance because charges now have a greater length to move through.

Parallel Resistors: As mentioned briefly in previous chapters, resistors are in parallel only if they share the same end nodes. Some examples include fig. 22, R_3 and R_4 in fig. 26, and (less obviously perhaps) R_1, R_2, R_5, and R_3 in fig. 38. Adding more resistors in parallel *always decreases* the effective resistance, since the current now has more paths through which to flow. That is, the effective area has increased, and by eq. (30), an increase in area leads to a decrease in resistance.

The formula to simplify 2 parallel resistors is:

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \equiv R_1 \parallel R_2 \tag{40}$$

We often use the second notation as a shorthand called the "parallel operator" since parallel resistances come up quite often. It is worth noting that $x \parallel y$ *does not mean* that the elements x and y are in parallel for all circuit elements. Also, the formula for 3 parallel resistors is not simply $\frac{\text{product}}{\text{sum}}$; the best way to simplify many parallel resistors is to combine in pairs using eq. (40).

See fig. 51 for an example simplifying a complex network! The terms were getting a bit lengthy toward the end, so the single-resistor answer is below for anyone who wants to check. Notably, the $R_1 \parallel R_2$ term is actually itself in parallel with the left sub-network, from the perspective of a and b!

$$R_{eq} = \underbrace{\left[\underbrace{\left((R_3 \parallel R_4) + R_5 \right)} + \underbrace{\left((R_6 + R_7 + R_8) \parallel R_{10} \right)} + R_9 \right]} \parallel \underbrace{[R_1 \parallel R_2]}$$

Problems

✠ **Problem:** Is it *true* or *false* that for fig. 52, $V_{ab} < V_S$ for all positive (practical) R_1 and R_2?

Answer: True

Quick Solution: The answer follows directly from the voltage divider equation, eq. (36). Since the denominator is always larger than the numerator for all positive resistances, the statement is true. The quantity $\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2}$ is called the gain of the circuit (and for the divider, the gain is always < 1).[89]

✠ **Problem:** Given fig. 53, find an expression for R_{eq} between a and b.

Answer: $(R_1 \parallel R_2) \parallel (R_3 + R_5)$
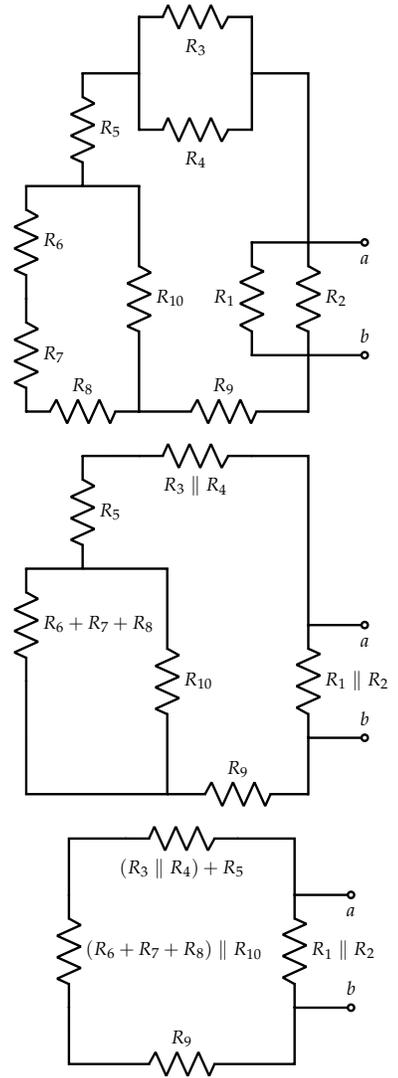
Figure 51: Sequence of resistor network simplifications using eq. (39) and eq. (40).
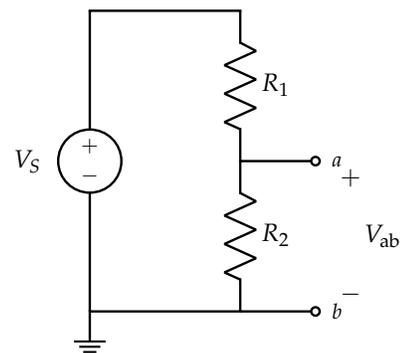
Figure 52: Voltage Divider with both output terminals labeled.

[89] If one wanted to make the argument for an infinite R_2 (which is definitely impractical), they'd still need to account for the fact that no current can flow in this circuit so no voltage can be dropped, since R_2 resembles an open circuit.

Quick Solution: Combine R_3 and R_5 in series, R_1 and R_2 in parallel, and ignore R_4 since it's shorted. Then combine $R_3 + R_5$ in parallel with $R_1 \parallel R_2$. See the LONG SOLUTION for details and explanations!

Long Solution: Our approach to simplify resistor networks will be to apply series and parallel simplifications where possible. First, note that R_3 and R_5 have the same current through them, by KCL at the node between. So, they are in series, and can be represented with an equivalent resistor $R_3 + R_5$. Also, R_1 and R_2 share terminal nodes a and b, so they are in parallel. We combine them into a single equivalent resistor $R_1 \parallel R_2$. Next, we notice that R_4 has both ends connected to the same node, so it is shorted. Whenever a resistor is shorted in this way, it can be represented as an open circuit. Why? Well, a nonzero resistor has infinitely more resistance than a short, so it may as well be an open circuit compared to the other path, and we can basically forget about it. Our circuit now looks like fig. 54.

In this form, it becomes clear that $R_1 \parallel R_2$ and $R_3 + R_5$ are in parallel, and we combine them using the parallel operator, arriving at $(R_1 \parallel R_2) \parallel (R_3 + R_5)$.

✠ **Problem:** Is it *true* or *false* that given a circuit with power consumption P, its thevenin equivalent also has consumption P?

Answer: False

Quick Solution: Remember that the I-V characteristic looking between 2 terminals is the *only* thing preserved by an equivalent circuit, and power is not necessarily conserved. Consider the counterexample in fig. 55, where the original circuit dissipates $P = \frac{V_S^2}{R}$ but thevenin equivalent dissipates no power (here, for 2 reasons: the equivalent resistance is 0Ω and more importantly, the circuit is not a closed loop so no current can flow).

✠ **Problem:** Given fig. 56, find R_{th} and V_{th} between a and b.

Answer: $R_{\text{th}} = 9\Omega$, $V_{\text{th}} = 8V$

Long Solution: We follow the steps outlined earlier and find V_{th} and R_{th} directly. To find V_{th}: note that NVA (or application of the voltage divider equation, eq. (36)) yields the potentials at the nodes above and below the 6Ω resistor: 8V above and 0V below. So $V_{\text{th}} = V_{\text{top}} - V_{\text{bottom}} = 8V$. The 7Ω resistor is "dangling" from the perspective of the circuit, and no current can flow through it because there is only an open node a on the other end. If any current flowed through this resistor, charge would be accumulating at a or being generated at a, and neither can happen. This means there can be no voltage drop across that resistor, so $V_a \equiv V_{\text{top}}$.

To find R_{th}: we have to zero out the independent voltage source. The voltage source becomes a wire. Then, the 3Ω and 6Ω share terminal nodes due to the short, and they are in parallel. Using eq. (40) to simplify: $R_{\text{eq}} = \frac{3 \cdot 6}{3 + 6} = 2\Omega$. Now, this equivalent resistance is in series with the 7Ω resistor by KCL. So we apply eq. (39) and add them; $R_{\text{th}} = 9\Omega$.
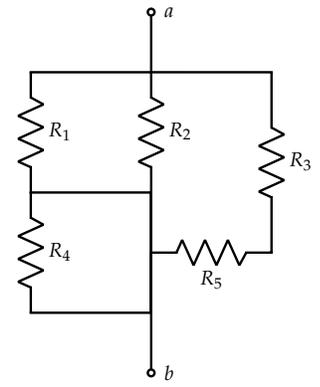
Figure 53: Resistor network to find the equivalence for, between terminals a and b.
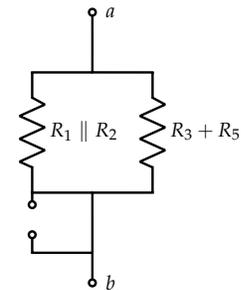
Figure 54: Intermediate step to simplify the resistor network in fig. 53.
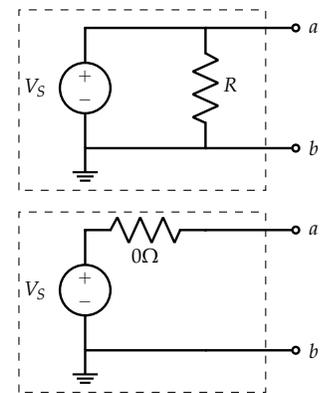
Figure 55: The power consumption of the thevenin equivalent is not the same as the original circuit. Note that here, the equivalent resistance of the circuit is 0Ω; the voltage source, when zeroed out, acts like a short so there's a no-resistance path between a and b.
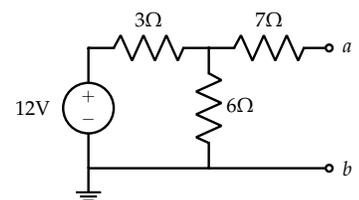
Figure 56: Circuit to find R_{th} and V_{th} for given terminals a and b.

There is another approach we can take to find R_{eq}: $\frac{V_{th}}{I_N}$. We already found V_{th} above, and to find I_N, we form a short-circuit between a and b and find the resulting current.[90] Performing NVA is one way to find this current and will certainly work, but we can try something different; we simplify the resistor network down to a single equivalent resistor to find the current draw out of the voltage source. Then, we notice that a portion of this current goes into the 7Ω branch and the remainder goes down into the 6Ω branch.

First, the equivalent resistance; if there is a short between a and b, then the 6Ω and 7Ω resistors will be in parallel (see fig. 57), and $6 \parallel 7 = \frac{42}{13}$. Then, this equivalent resistor is in series with the 3Ω resistor, so the total is $3 + \frac{42}{13} = \frac{81}{13}\Omega$. Using this value and Ohm's Law (eq. (31)), we find I_S coming out of the 12V source; $I_S = \frac{12}{81/13} = \frac{12 \cdot 13}{81} = \frac{156}{81}$A.

Now, we track this current through the original circuit; all of I_S goes through the 3Ω resistor, but splits at the node to its right. Applying the current divider equation eq. (34), the current into the 7Ω resistor is $I_{7\Omega} = I_S \frac{6}{6+7} = \frac{8}{9}$A. Calculating R_{th} : $\frac{V_{th}}{I_N} = \frac{8}{8/9} = 9\Omega$. The results match as they should!
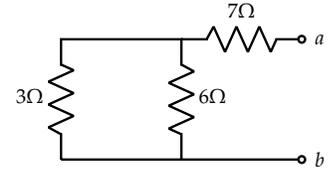
Figure 57: Network with voltage source zeroed, which can be used to find equivalent resistance between a and b.

✠ **Problem:** Given fig. 58, find R_{th} and V_{th} between a and b.

Answer: $R_{th} = 10\Omega$, $V_{th} = 16$V

Long Solution: Note the similarity to the previous question, but there is now an additional independent current source! Note that if we take the approach of network simplifications to find R_{eq}, this won't really affect our approach since we will zero all independent sources anyways, and a zeroed current source is an open circuit. However, when it comes to calculating V_{th}, the principle of superposition will prove quite useful.

Figure 58: Circuit to find R_{th} and V_{th} for given terminals a and b.

We first find the node potential contributions from the voltage source; this is exactly as in the previous problem, so following the same steps, $V_{th, \text{10V}} = 8$V. Next, to find the contribution from the current source, we can perform NVA rigorously, or apply the current divider equation and recognize that the potential drop across *either* the 5Ω resistor or the 20Ω resistor gives $V_{th, \text{2A}}$. The current through the 20Ω branch is $I_{20\Omega} = I_S \cdot \frac{5}{5+20} = 2 \cdot 0.2 = 0.4$A. Then the voltage drop $V_{20\Omega} = I_{20\Omega} \cdot 20\Omega = 8$V. Adding the contributions from both sources: $V_{th, \text{total}} = 8 + 8 = 16$V.

Figure 59: Network with both independent sources zeroed, which can be used to find equivalent resistance between a and b.

Now, to find R_{th}, if we want to use resistor network simplifications, then following the exact same steps as the previous problem (see fig. 59), $R_{th} = (5 \parallel 20) + 6 = 4 + 6 = 10\Omega$.

As we've seen, we can also approach this task by finding I_N, the short-circuit current between a and b. Here, there are two sources to consider, so it makes sense to use superposition, To find the current contribution from the voltage source, we repeat the same steps as above, yielding: $I_{N,\text{10V}} = 0.8$A. For the current source, we can take a similar approach. First, zeroing out the voltage source leaves a short-circuit in its place. The 2A current then divides between 5Ω, 6Ω, and

20Ω and we want the current through the 6Ω branch. So we can combine the 5Ω and 20Ω resistors and applying the current divider equation eq. (34), find:
$I_{N,2A} = 2 \cdot \frac{5\|20}{6+(5\|20)} = 2 \cdot \frac{4}{10} = 0.8A.$

Adding the contributions: $I_{N,\text{ total}} = 1.6A$. So $R_{\text{th}} : \frac{V_{\text{th}}}{I_{N,\text{total}}} = \frac{16}{1.6} = 10\Omega$. Once again, our answers match! There are takeaways from these two problems; some approaches tend to be much faster than others, and also less error-prone, but the more kinds of circuits we see, the easier the process becomes.

⌘ **Problem:** Is it *true* or *false* that combining (positive) resistors in parallel yields an equivalent resistance that is less than any of the individual resistances?

Answer: True

Quick Solution: The result follows from rearranging eq. (40) into $R_1 \| R_2 = R_2 \frac{1}{1+\frac{R_2}{R_1}}$, such that all individual resistors will be larger than the equivalent resistor. We can inductively apply this reasoning to any number of resistors.

Long Solution: To simplify many resistors in parallel, we apply eq. (40) repeatedly; that is, if we can show that $R_1 \| R_2 < R_1, R_2$, we will have shown that the statement is true since we can apply the results inductively throughout all simplifications. So: $R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2} = \frac{R_1}{\frac{R_1+R_2}{R_2}} = R_1 \frac{1}{1+\frac{R_1}{R_2}}$. Similarly: $R_1 \| R_2 = R_2 \frac{1}{1+\frac{R_2}{R_1}}$. Notice that in each case result of the parallel expression is to divide one of the resistances by $1 + \frac{R_2}{R_1}$, which is always > 1 for positive resistances. We've shown the statement holds for 2 resistors; applying the same logic pairwise for any k resistors, the result still holds. This can be a very useful sanity check on exams!

Intuitively, consider that adding a resistor in parallel amounts to increasing the cross-sectional area across which current can flow, thereby reducing the effective resistance to movement of charge.

⌘ **Problem:** Given fig. 60, what is V_{ab}?

Figure 60: Circuit to find V_{ab} (and later, R_{th}) for.

Answer: $V_{ab} \approx 8.57V$

Quick Solution: We can apply superposition. Considering only the voltage source, with the current source zeroed out (open circuit), we have a simple voltage divider; the potential at a is $V_a = 10 \cdot \frac{5}{5+2} \approx 7.143V$. Considering only the current source (voltage source becomes a short) and applying the current divider equation (eq. (34)), $V_a = (1A \cdot \frac{2}{2+5}) \cdot 5\Omega \approx 1.428V$. In both cases, b is grounded. So adding the contributions: $V_{ab} = 7.14 + 1.43 \approx 8.57V$.

⌘ **Problem:** For the same circuit, fig. 60, what is R_{th}, with terminals a and b?

Figure 61: This redrawn form makes it easier to find the thevenin resistance!

Answer: $R_{\text{th}} = \frac{10}{7}\Omega$

Quick Solution: It can help to redraw the circuit as in fig. 61, with the independent sources appropriately zeroed out. Now, it can be seen that the 2Ω and 5Ω resistors are in parallel: $R_{\text{th}} = (2 \| 5)\Omega = \frac{10}{7}\Omega$.

Practice Set 8: Capacitors

Relevant Equations/Information

Capacitor Structure and Physics: A typical capacitor is a structure composed of two parallel metal plates, and therefore has a circuit symbol as given in fig. 62. [91]

We can draw some analogous connections between resistors and capacitors. For our purposes, capacitors do not have polarity; their orientation doesn't impact their behavior. A capacitor has capacitance, and the units of capacitance are Farads, ($F = \frac{C}{V}$). Capacitance depends on the physical geometry of a capacitor. Observe fig. 63; given such a pair of metal plates with a certain area A and separation between then d, the capacitance is given by eq. (41), where ϵ is a constant with value $8.854 * 10^{-12} \frac{F}{m}$, the permittivity of free space.

$$C = \epsilon \frac{A}{d} \tag{41}$$

The function of a parallel plate capacitor is to store charge on its plates. The amount of charge Q on a capacitor (magnitude of charge on *each* plate) is related to its geometrical structure (capacitance C) and the voltage applied to it V, as indicated by eq. (42).

$$Q = CV \tag{42}$$

Let's examine these equations intuitively; when we apply a voltage across the conductive plates, we create a potential difference, and so charges will build up on the plates as shown in fig. 63.[92] But they will not build up indefinitely because this potential difference is not infinitely strong. At some point, an additional positive charge will be ambivalent about joining the plate; while there is a potential difference $V_+ - V_-$ between the plates attracting the charge, there are also repulsive forces from the charges that already exist on the plate. After this amount of critical charge forms on the plate, no additional charge will enter.

The more the area, the more the total charge that can fit on the plate because the individual charges can spread out more, decreasing the repulsive forces. But the smaller the distance between the plates, the more strongly the $+$ and $-$ charges
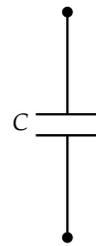
Figure 62: The circuit symbol for a capacitor. Note that the two horizontal lines represent the capacitor's metal plates.
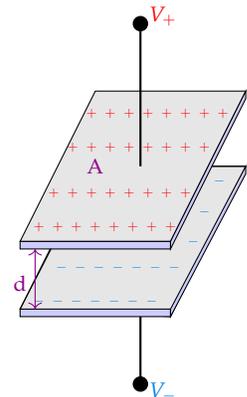
Figure 63: A physical diagram of a capacitor, with 2 metal plates and an insulator (called a dielectric, commonly just air) between them.

[92] These charges must be equal in magnitude and opposite in sign by the principles of conservation of charge. We had a neutral system before applying the voltage, so the system overall must be neutral afterward also.

attract each other (recall electrostatic forces)! That is, for the same voltage, decreasing the distance of plate separation increases the charge that will build up on the plate. This intuition matches the equations! The best capacitors will maximize A and minimize d.[93]

Capacitor Energy Storage: When we apply a potential difference, charges build up on the plates; these charges are the reason that a capacitor stores energy. The repulsion between charges on a given plate means that moving a charge onto the plate takes energy (supplied by the voltage source). The more charge already on a plate, the more energy it takes to push another charge on. When we derived the formula for power, we used the fact that voltage is potential energy per unit charge: $V = \frac{dE}{dQ}$. We use the same starting point and rearrange, substitute, and integrate[94]:

[94] The separation of variables technique is not necessary to understand for this class.

$$V = \frac{dE}{dQ}$$

$$dE = V dQ$$

$$dE = V \cdot d(CV)$$

$$\int_0^E dE = C \int_0^V V \cdot dV$$

The result is:

$$E = \frac{1}{2}CV^2 \tag{43}$$

This formula gives the energy of a capacitor when it is fully charged, holding the complete Q determined by C and V.

Capacitor I-V Relationship and Behavior:

From eq. (42), we can derive the I-V relationship for a capacitor by differentiating both sides with respect to time, and we arrive at eq. (44).

$$I = C\frac{dV}{dt} \tag{44}$$

There is a very important consequence to this equation; current only flows through a capacitor when the voltage across it is *actively changing*. If the voltage across it is constant, then the plates are already full of charge for that voltage; any additional charge will feel the repulsive forces of the existing charges and will not want to enter the plate.

We haven't yet explained how current physically flows in a circuit with a capacitor. It was simpler with a resistor, because charges could move through the

resistor. But here, there is a physical gap between the plates, so charges can't move across. What happens? Note that the negative charges build up on the bottom plate in fig. 63; this happens because the positive charges on the top plate push the positive charges away from the bottom plate. This means that in reality, the charges entering the top plate are not the same exact physical charges that exit the bottom plate, but it doesn't matter! From the perspective of the other circuit elements (and for our purposes), current flows just the same.

Figure 64: A capacitor will fully charge up to $Q = CV_S$ almost immediately if connected to a voltage source! Current will flow until a potential difference of V_S is established across the plates of the capacitor, by the charges.

Let's look at a couple examples; suppose we have a capacitor sitting there, as in fig. 62, and we connect it to a voltage source, as in fig. 64. The capacitor will become fully charged nearly instantaneously because we assume that there is no limit to the current that can flow in a circuit.[95] Charge will flow very quickly onto the plates (current flows because the voltage changes from 0 to V_S) until a potential difference of V_S is formed across the plates. After charge $Q = CV_S$ is on each plate, no additional charges will enter the plates.

What happens if we instead connect an ideal constant current source to a capacitor, as in fig. 65?[96] The current source will force current to flow in the circuit, and therefore, the voltage drop across the capacitor will increase linearly with time. Why linearly? We can manipulate eq. (44) as follows:[97]

$$I = C\frac{dV}{dt}$$

$$\int_0^t I \cdot dt = C \int_0^V dV$$

$$It = C(V(t) - V(0))$$

$$V(t) = \frac{I}{C}t + V(0)$$

Figure 65: A capacitor will continue to charge up more and more over time if connected to a constant current source!

[95] For anyone who's studied RC circuits or similar (not required), a resistor will act as a current-limiter, and sets the charging time constant of the filter.

[96] You'd explode your capacitor if you did this.

[97] Here, $V(0)$ is the initial voltage across the capacitor, before the current source is connected. This term arises from the fact that integration introduces a constant of integration, which is $V(0)$ here. This will often, but not always, simply be 0V.

Notice that the slope of voltage with respect to time is linear, with slope $\frac{I}{C}$. This makes intuitive sense; if the current source is forcing more current (more charge) onto the plates, then the voltage across the plates will increase more quickly. But if the capacitance is larger, then the capacitor can hold more charge so the voltage drop won't increase as quickly.

Capacitor Network Simplifications: Just as with resistors, we can make capacitor network simplifications for parallel and series connections,[98] but there are some important differences. We present geometrical intuition below; the course notes derive the same results using test "changes-in-voltages" and currents.

[98] These terms are defined the same as they were for resistors; parallel capacitors share terminal nodes, series capacitors have the same current through them by KCL.

Series Capacitors: Observe fig. 66; the equivalent capacitor effectively has a larger separation distance between the top-most plate of C_1 and the bottom-most plate of C_2; the result is that the equivalent capacitance is smaller than either individual capacitance, just like for parallel resistors. The middle two plates actually carry equal and opposite amounts of charge, so their net contribution to charge storage is 0; the presence of more plates doesn't allow for more charge storage, and the only result is to increase the effective d. After performing NVA, we find a formula similar to that for parallel resistors:

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \equiv C_1 \parallel C_2 \tag{45}$$

We could also arrive at eq. (45) by noting that from fig. 67, combining the capacitors effectively makes the distance $d_{C_1} + d_{C_2}$, such that $C_{eq} = \epsilon \frac{A}{d_{C_1} + d_{C_2}} \implies \frac{1}{C_{eq}} = \frac{1}{\epsilon \frac{A}{d_{C_1} + d_{C_2}}} = \frac{d_{C_1} + d_{C_2}}{\epsilon A} = \frac{1}{\frac{\epsilon A}{d_{C_1}}} + \frac{1}{\frac{\epsilon A}{d_{C_1 2}}} \implies \frac{1}{C_{eq}} = \frac{1}{\frac{\epsilon A}{d_{C_1}}} + \frac{1}{\frac{\epsilon A}{d_{C_1 2}}}$. We rearrange and arrive at eq. (45).

Now, we must address a point of common confusion; the parallel operator ($\parallel$) is just a mathematical tool; it happened to describe the equivalence for resistors in parallel but the operator actually applies to capacitors in series. As with resistors in parallel, capacitors in series are best simplified pairwise.

Figure 66: Capacitors in series.

Parallel Capacitors: Geometrically, capacitors in parallel have a larger effective area (visually, we see this in fig. 67), because we can think of their areas as being combined from the perspective of charge that builds on the plates. This means that combining parallel capacitors is the same as adding them:

$$C_{eq} = C_1 + C_2 + \ldots C_k \tag{46}$$
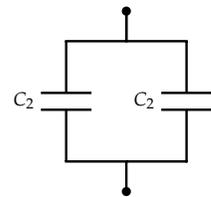
Figure 67: Capacitors in parallel.

This is easier to derive; since the area term is in the numerator, we have $C_{eq} = \epsilon \frac{A_{C_1} + A_{C_2} + \ldots + A_{C_k}}{d} = \epsilon \frac{A_{C_1}}{d} + \epsilon \frac{A_{C_2}}{d} + \ldots \epsilon \frac{A_{C_k}}{d} = C_1 + C_2 + \ldots + C_k$.

Problems

✠ **Problem:** Given a parallel plate capacitor, to maximize its capacitance, do you want to increase or decrease plate area A? Plate separation distance d? Insulator permittivity ϵ?

Answer: Increase: ϵ, A, Decrease: d

Quick Solution: The answer follows directly from eq. (41). Area and permittivity

are in the numerator of the capacitance formula, and separation distance is in the denominator.

✠ **Problem:** Is it *true* or *false* that combining (positive) capacitors in series yields an equivalent capacitance that is less than any of the individual capacitances?

Answer: True

Quick Solution: Just as we did for resistors in the previous chapter, we can manipulate eq. (45): $C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \equiv C_1 \frac{1}{1 + \frac{C_1}{C_2}} \equiv C_2 \frac{1}{1 + \frac{C_2}{C_1}}$. Notice that we divide each individual capacitance by a value larger than 1, so the net capacitance decreases for series networks.

✠ **Problem:** Is it *true* or *false* that combining (positive) capacitors in parallel yields an equivalent capacitance that is less than any of the individual capacitances?

Answer: False

Quick Solution: Inspecting eq. (46) we see that capacitors in parallel add (the area term is in the numerator, opposite of a resistor). Therefore, adding capacitors in parallel increases effective capacitance; for the same voltage, combining capacitors allows for greater charge storage as compared to any one capacitor.

✠ **Problem:** Is it *true* or *false* that if we know $I(t)$ for a capacitor, we can find $V(t)$ for any t?

Answer: False

Quick Solution: While we can theoretically perform the integration of current over time for any arbitrary current function and solve for the change in voltage, we would still need the initial condition in order to precisely find $V(t)$. Without $V(0)$, we can only find $\Delta V = V(t) - V(0)$, not the value of $V(t)$ itself.

✠ **Problem:** Given fig. 68, what is $V_{out}(t)$ in terms of the given circuit element values? Assume that for all capacitors, $V_{C_i}(0) = 0$ (initially uncharged).

Answer: $\frac{C_3 I_S}{C_2 C_3 + C_2 C_4 + C_3 C_4} t$

Long Solution: There are two main approaches we can take to solve this question: NVA (applications of KCL) and Capacitor Network Equivalences. Both of them require some labeling of the given circuit, as in fig. 69, and a good amount of algebra. It's good practice! I present both approaches below:

KCL: We start by writing the following KCL equations:

$$u_0 : I_S = I_{C_1}$$
$$u_1 : I_{C_1} = I_{C_2} + I_{C_3}$$
$$u_2 : I_{C_3} = I_{C_4}$$
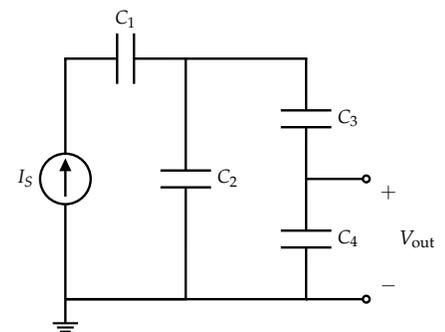
Figure 68: Diagram for the problem to the left.
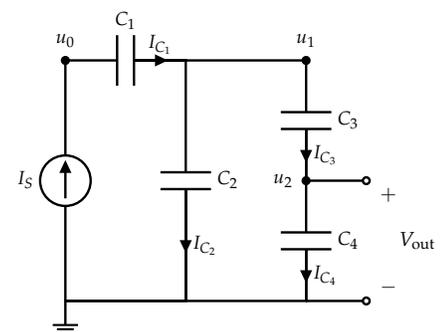
Figure 69: Labeled diagram to prepare for solving the problem.

Recalling eq. (44) as the current through a capacitor, we substitute:

$$u_0 : I_S = C_1 \frac{d(u_0 - u_1)}{dt}$$

$$u_1 : C_1 \frac{d(u_0 - u_1)}{dt} = C_2 \frac{d(u_1 - 0)}{dt} + C_3 \frac{d(u_1 - u_2)}{dt}$$

$$u_2 : C_3 \frac{d(u_1 - u_2)}{dt} = C_4 \frac{d(u_2 - 0)}{dt}$$

Notice that we can plug the result of the first equation directly into the second and get $I_S = C_2 \frac{d(u_1 - 0)}{dt} + C_3 \frac{d(u_1 - u_2)}{dt}$; the presence of C_1 (and its value) doesn't at all impact the current entering C_2, C_3, or C_4. The goal is to find V_{out}, but we must first find $\frac{dV_{\text{out}}}{dt} \equiv \frac{d}{dt} u_2$, which we can then integrate over time. Fortunately, we have this derivative in our third equation; let's isolate and solve:

$$C_3 \frac{d(u_1 - u_2)}{dt} = C_4 \frac{d(u_2 - 0)}{dt}$$

$$C_3 \frac{du_1}{dt} = C_3 \frac{du_2}{dt} + C_4 \frac{du_2}{dt}$$

$$\frac{du_2}{dt} = \frac{C_3}{C_3 + C_4} \frac{du_1}{dt}$$

Now, we plug into the second equation to get rid of the dependency on $\frac{du_2}{dt}$:[99]

$$I_S = C_2 \frac{d(u_1 - 0)}{dt} + C_3 \frac{d(u_1 - u_2)}{dt}$$

$$I_S = C_2 \frac{du_1}{dt} + C_3 \frac{du_1}{dt} - C_3 \frac{du_2}{dt}$$

$$I_S = C_2 \frac{du_1}{dt} + C_3 \frac{du_1}{dt} - C_3 \left(\frac{C_3}{C_3 + C_4} \frac{du_1}{dt} \right)$$

Solving for our only unknown, $\frac{du_1}{dt}$:[100]

$$\frac{du_1}{dt} = I_S \frac{C_3 + C_4}{C_2 C_3 + C_2 C_4 + C_3 C_4}$$

Now we go back to our relationship between $\frac{du_2}{dt}$ and $\frac{du_1}{dt}$; plugging in:

$$\frac{dV_{\text{out}}}{dt} \equiv \frac{du_2}{dt} = I_S \frac{C_3}{C_2 C_3 + C_2 C_4 + C_3 C_4}$$

Integrating both sides with respect to t by separating variables (and noting the initial uncharged condition) yields:[101]

$$V_{\text{out}}(t) = \frac{C_3 I_S t}{C_2 C_3 + C_2 C_4 + C_3 C_4}$$

Capacitor Equivalences: We can also find u_1 directly using equivalent capacitances. Combining C_2, C_3 and C_4 into an equivalent capacitor C_{eq} yields C_1 in series with C_{eq} and u_1 in between. Both have current I_S flowing through them. If we can solve for u_1, then we can apply the capacitive voltage divider formula (derived below) to find V_{out}.

Let's proceed by using eq. (44) to find $\frac{du_1}{dt}$. First, finding C_{eq}:

$$C_{\text{eq}} = (C_3 \parallel C_4) + C_2$$

$$= \frac{C_3 C_4}{C_3 + C_4} + C_2$$

[99] If you're wondering why we don't simply solve the second equation for $\frac{du_1}{dt}$ in terms of $\frac{du_2}{dt}$ and plug back in, it's because the presence of the I_S terms makes the equation affine and the algebra becomes much harder than anticipated; it's better to do it this way!

[100] Steps for completeness:

$$I_S = (C_2 + C_3) \frac{du_1}{dt} - \frac{C_3^2}{C_3 + C_4} \frac{du_1}{dt}$$

$$I_S = \frac{du_1}{dt} \left((C_2 + C_3) - \frac{C_3^2}{C_3 + C_4} \right)$$

$$\frac{du_1}{dt} = I_S \left(\frac{C_2(C_3 + C_4) + C_3(C_3 + C_4) - C_3^2}{C_3 + C_4} \right)^{-1}$$

$$\frac{du_1}{dt} = \frac{I_S(C_3 + C_4)}{C_2(C_3 + C_4) + C_3(C_3 + C_4) - C_3^2}$$

$$\frac{du_1}{dt} = I_S \frac{C_3 + C_4}{C_2 C_3 + C_2 C_4 + C_3 C_4}$$

[101] Steps for completeness:

$$dV_{\text{out}} = I_S \frac{C_3}{C_2 C_3 + C_2 C_4 + C_3 C_4} dt$$

$$\int_0^{V_{\text{out}}} dV_{\text{out}} = I_S \frac{C_3}{C_2 C_3 + C_2 C_4 + C_3 C_4} \int_0^t dt$$

$$V_{\text{out}} = I_S \frac{C_3}{C_2 C_3 + C_2 C_4 + C_3 C_4} t + \underbrace{V_{\text{out}}(0)}_{0}$$

$$= \frac{C_2(C_3 + C_4) + C_3 C_4}{C_3 + C_4}$$

Now, we solve for $\frac{du_1}{dt}$:

$$I_S = C_{eq} \frac{du_1}{dt}$$

$$\frac{du_1}{dt} = I_S \frac{C_3 + C_4}{C_2(C_3 + C_4) + C_3 C_4}$$

The only question now is, how does $\frac{du_1}{dt}$ relate to $\frac{du_2}{dt} \equiv \frac{dV_{out}}{dt}$? It's time to derive the capacitor-based voltage divider equation.

Given the circuit in fig. 70, what is $\frac{dV_{out}}{dt}$ in terms of $\frac{dV_S}{dt}$? Once we solve this, we can directly apply the results to our original problem. Applying KCL and eq. (44), $I_{C_1} = I_{C_2} \implies C_1 \frac{d(V_S - V_{out})}{dt} = C_2 \frac{dV_{out}}{dt} \implies \frac{dV_{out}}{dt} = \frac{C_1}{C_1 + C_2} \frac{dV_S}{dt}$. Substituting appropriately into our problem:

$$\frac{dV_{out}}{dt} = \frac{C_3}{C_3 + C_4} \left(I_S \frac{C_3 + C_4}{C_2(C_3 + C_4) + C_3 C_4} \right)$$

$$\frac{dV_{out}}{dt} = I_S \frac{C_3}{C_2(C_3 + C_4) + C_3 C_4}$$

To conclude the problem, we integrate using separation of variables, noting the initial uncharged state of C_4 (that is, $V_{C_4} \equiv V_{out} = 0$):

$$V_{out}(t) = \frac{C_3 I_S t}{C_2 C_3 + C_2 C_4 + C_3 C_4}$$

Note an important overall result; since we only care about changes in voltage, we can ignore C_1 since the current source sets the current division at u_1.

�҂ **Problem:** Given fig. 71, find C_{eq} between terminals a and b.

Answer: $3C$

Quick Solution: This problem is similar to the one based off fig. 57; we apply similar techniques, but for capacitors. We first notice that the $2C$ capacitors are in series, and so can be simplified with eq. (45): $2C \parallel 2C = C$. Also, the middle-top capacitor is shorted out by the wire on the top-right, so it is effectively an open circuit. Finally, the remaining C capacitors are in parallel since they share terminal nodes; $C + C = 2C$. After plugging in this first round of simplifications, we have fig. 72. We simplify the last parallel capacitors by adding them, ignore the short-circuited capacitor, and arrive at $2C + C = 3C$.

✄ **Problem:** Given fig. 73, what is the total energy stored in all the capacitors?

Answer: $\left(\frac{C_1}{2} + \frac{C_2}{4} \right) V_S^2$

Quick Solution: Using eq. (43), we can take one of two approaches; find the voltage drop across each capacitor individually and add the individual energy storages, or simplify the network to an equivalent capacitor and apply the equation once.

Figure 70: Capacitor-based voltage divider.
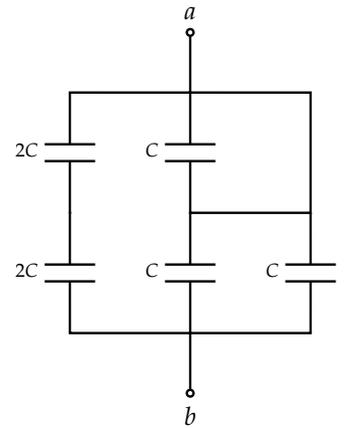
Figure 71: Capacitor network to find equivalence for, between terminals a and b.

Figure 72: Intermediate step to simplify the network in fig. 71.
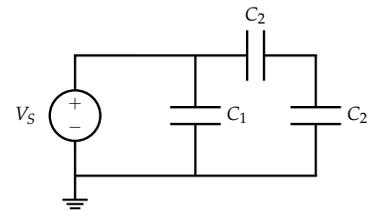
Figure 73: Circuit to find capacitor energy storage for for.

Individual Energy Storages: By symmetry in the capacitor voltage divider (discussed in more detail in a previous problem), each C_2 capacitor has a $\frac{V_S}{2}$ voltage drop across it, so each stores energy $E_{C_2} = \frac{1}{2}C_2\left(\frac{V_S}{2}\right)^2$. Applying KVL with V_S and C_1, this C_1 capacitor has a full V_S voltage drop across it, so it stores $E_{C_1} = \frac{1}{2}C_1V_S^2$. Adding them all up: $E_{\text{tot}} = 2E_{C_2} + E_{C_1} = \frac{C_2V_S^2}{4} + \frac{C_1V_S^2}{2} = \left(\frac{C_1}{2} + \frac{C_2}{4}\right)V_S^2$.

Equivalent Capacitor's Energy Storage: By applying series and parallel simplifications, we find that from the perspective of the voltage source, the equivalent load capacitance is $(C_2 \parallel C_2) + C_1 = C_1 + \frac{C_2}{2}$. This equivalent capacitor has the full V_S dropped across it; $E_{\text{tot}} = \frac{1}{2}\left(C_1 + \frac{C_2}{2}\right)V_S^2 = \left(\frac{C_1}{2} + \frac{C_2}{4}\right)V_S^2$.

Practice Set 9: Op-Amps and Golden Rules

Figure 74: Symbol of an op-amp (operational amplifier). V_{DD} represents the maximum voltage that this op-amp can output and V_{SS} represents the minimum; if not specified, it is often implicit that there is no such limit.

Relevant Equations/Information

Capacitive Touchscreen, Charge Sharing, Switches, Measuring Capacitance: Just as with the 2D Resistive Touchscreen, the notes are a good resource to learn these concepts. They aren't covered in the practice sets, but are still important!

Op-Amp Basic Model and Purpose: Op-Amp stands for *operational amplifier*, and it amplifies a small input voltage signal into a much larger output voltage. The circuit symbol for an op-amp is shown in fig. 74; the output, V_{out}, is an amplified version of the *difference* between the inputs, V_+ and V_-.[102] This difference is often referred to as the signal voltage, and the amplification factor is a unit-less quantity A, also referred to as the signal gain.

[102] I will sometimes refer to V_- as the *inverting input*, and V_+ as the *non-inverting input*. This is standard terminology, and I will use these interchangeably with V_- and V_+.

The most general form of the output voltage of an op-amp is:

$$V_{\text{out, ideal}} = \underbrace{\frac{V_{DD} + V_{SS}}{2}}_{\text{rail-determined offset}} + \underbrace{A\left(V_+ - V_-\right)}_{\text{signal gain}} \qquad (47)$$

In many cases, the V_{DD} high rail and V_{SS} low rail have the same magnitude (and opposite sign), so the bottom VCVS representing the rail-offset (see fig. 75) is simply 0V. In this case, when the input voltages are the same ($V_+ - V_- = 0$), the output lies exactly at 0V.
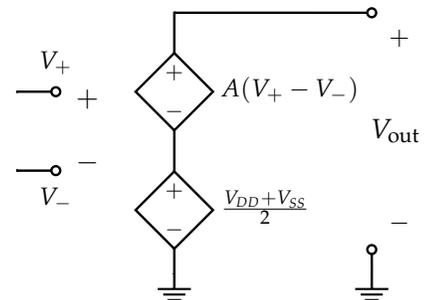
Figure 75: Simplified internal model for an op-amp. Note that there are two primary determinants of the output voltage; the rails (which become averaged) and the difference in the input voltages (which become amplified). Be sure to see eq. (47) and recognize how the 2 terms connect to this diagram.

An op-amp can only output a voltage in the range $[V_{SS}, V_{DD}]$; if the ideal output signal lies outside of this range, the actual output will be "clipped" to lie in the given constrained range (as in the second and third cases below). Simply put, the actual output voltage is a piecewise-linear function of the ideal output voltage (eq. (47)) and the rails (use fig. 76 as a visual aid):

$$\text{if } V_{\text{out, ideal}} < V_{SS} : V_{\text{out, actual}} = V_{SS} \text{ [railing]}$$

$$\text{if } V_{SS} \leq V_{\text{out, ideal}} \leq V_{DD} : V_{\text{out, actual}} = V_{\text{out, ideal}}$$

$$\text{if } V_{\text{out, ideal}} > V_{DD} : V_{\text{out, actual}} = V_{DD} \text{ [railing]}$$

Op-Amp as a Comparator: An op-amp can also be used to compare two input voltages, and the sign of the output will indicate which input voltage is larger. Suppose we have some known reference voltage V_{ref} as V_- (that is, at the inverting input) and some other voltage $V_?$ as V_+ (at the non-inverting input). We often want to know whether $V_?$ is smaller or larger than V_{ref}.

The gain A of an op-amp is quite high (on the order of 10^6 or more), and in this class, we often treat it as infinite. This means that the "linear" region where the output is an un-clipped, factor-of-A amplified version of the input actually doesn't exist, or is very narrow. If the two input voltages are exactly equal, then the output is 0V but otherwise, the output immediately rails to V_{DD} (if $V_? > V_{\text{ref}}$) or rails to V_{SS} (if $V_? < V_{\text{ref}}$). Then, by observing the output, we can determine if $V_?$ is bigger or smaller than our reference voltage V_{ref}. A simple graph demonstrating this behavior is in fig. 76.

In practice, op-amps are highly complex internally because they need to work correctly in a variety situations, but the act of comparing two voltages is very simple. In situations when using an op-amp is overkill, we can instead use a dedicated comparator (circuit symbol in fig. 77), which also follows the behavior in fig. 76 (but with a perfectly vertical blue line, since it's ideal).

Negative Feedback (NFB):[103] At this point, one might be wondering how an op-amp can do anything *except* output a voltage at one of the rails. Isn't the internal gain so high that no matter what, any nonzero input voltage difference will be amplified to the point of clipping? Well, we can avoid this undesirable outcome using the concept of negative feedback.

Negative feedback refers to the idea that there is some output amount that is the ideal or intended amount; if the actual amount becomes larger than this reference, then the system must detect this deviation and bring it down to the reference. Similarly, if the actual amount drops too low, the system must bring it back up to the reference. For op-amps, negative feedback helps maintain the output voltage at a constant level *despite the fact* that the op-amp wants to rail the output. But how can the op-amp detect the deviation in the first place?

To incorporate negative feedback into our op-amp design, we must create a connection that goes from the output into the negative input, as shown in fig. 78. Let's
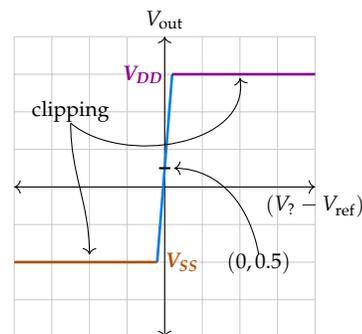
Figure 76: The blue region is the (super narrow, and in the ideal case, perfectly vertical) linear region where the input actually has "space" to be amplified without clipping, but note that for the vast majority of realistic inputs, the outputs sits at one of the rails (here, $V_{SS} \approx -2$V and $V_{DD} \approx 3$V). In this way, the sign of the output directly gives the relationship between $V_?$ and V_{ref}. Note that with zero input difference, the output voltage (y-intercept) sits at the rail-offset, $(0, \frac{V_{DD}+V_{SS}}{2}) = (0, 0.5)$.
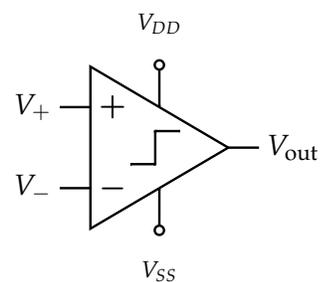
Figure 77: Circuit symbol for a comparator (note the similarity to the symbol for an op-amp).

[103] This section tends to cause great conceptual confusion, and might be worth reviewing a couple times to fully understand.

understand why this works; first, we assume the supplies are of equal magnitude, so the output lies at 0V when $V_+ = V_-$. Then, from the general op-amp behavior in eq. (47), we can say that $V_{out} = A(V_+ - V_-)$. But here, $V_- = V_{out}$! So what happens when we solve for V_{out}?

$$V_{out} = A(V_+ - V_-)$$

$$V_{out} = A(V_{in} - V_{out})$$

$$V_{out} + A \cdot V_{out} = A \cdot V_{in}$$

$$V_{out}(1 + A) = A \cdot V_{in}$$

$$V_{out} = V_{in}\frac{A}{A+1}$$

Interestingly, the output is a scaled version of the input. But remember how we said that in the ideal case, $A \to \infty$? If that's true, then the last equation above means $V_{out} = V_{in}$. The output is the same as the input![104] So clearly, the system has the ability to output a voltage that is between the rails (when V_{in} lies in $[V_{SS}, V_{DD}]$).

Checking for NFB: In fig. 78, we can see that the output is connected to the negative input terminal, but how can we confirm that the op-amp is in negative feedback? One convenient method is to check what happens if the output voltage happens to fluctuate by a little bit above the desired output; when this change occurs and propagates back to the input of the op-amp, does it cause the output to come back down to the desired level? If so, the system is in negative feedback.

For the buffer in fig. 78, if $V_{out} \uparrow$, then $V_- \uparrow$, and $V_+ - V_- \downarrow$ (the difference between the higher and lower voltages decreases if the lower voltage increases.) Then, $A(V_+ - V_-) \downarrow$ also. From this, we can directly say that $V_{out} \downarrow$. Notice how an increase in the output voltage ultimately led to a decrease in the output voltage. This is negative feedback at work, squashing any fluctuations to ensure the output remains at the expected level.[105]

The Buffer; A Practical NFB Example: When we present the fact that $A = \infty$ and $V_+ - V_- = 0$ in NFB, this seems to be wholly incompatible with the general op-amp equation in eq. (47) that $V_{out} = A(V_+ - V_-)$. Plugging in, we literally get $\infty \cdot 0$, which is undefined, so what's wrong? Indeed, the thing to keep in mind is that A is never *truly* infinite; this happens to be a convenient approximation, but for the mathematically inclined, let's use some actual numbers to convince ourselves that

[104] Note that even if A isn't infinite, so long as it is very large, $V_{out} \approx V_{in}$. If you're confused by the fact that there is somehow no input voltage difference but a nonzero voltage at the output, I recommend checking out the practical example a couple sections below.
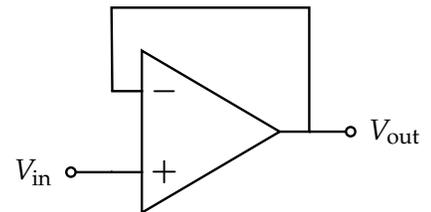
Figure 78: This example shows an op-amp wired in negative feedback. It is critical that the output is connected to the *negative* input, not the positive one. This configuration is also called a unity-gain buffer, and is used to isolate circuit blocks/stages, which will be covered later.

[105] It works the other direction too: if $V_{out} \downarrow$, then $V_- \downarrow$, $V_+ - V_- \uparrow$, $A(V_+ - V_-) \uparrow$ and finally, $V_{out} \uparrow$.

the concept and results derived using NFB aren't complete nonsense.

For the simplest analysis, we use fig. 78. Suppose $A = 1,000,000 = 10^6$, which is high but perfectly common. If $V_{in} = V_+ = 5V$, then V_-, same as V_{out} here, must be a precise value such that $10^6(5 - V_{out}) = V_{out}$. Solving, V_{out} isn't quite 5V; it's 4.999995V. See, the fact is, the resulting differential of about 5μV is enough for the op-amp to work with; it amplifies this difference into about 4.999995V at the output. In practical scenarios, this output is plenty close enough to $V_{in} = 5V$, which is why we feel comfortable saying $V_+ = V_-$ in NFB. If you encounter any equations of the form $\infty \cdot 0$ or similar and begin doubting your conceptual understanding, remind yourself of this practical example.

Figure 79: An op-amp with the currents feeding into the input terminals labeled. These currents will always be zero! There's no closed circuit inside the op-amp at the input, so the current cannot flow anywhere.

Op-Amp "Golden Rules": These are two very useful results to be familiar with, as they greatly simplify the analysis of op-amp configurations.

Golden Rule #1, *Zero Input Currents*: Observe fig. 79; $I_+ = I_- = 0$. That is, the currents going into the inputs of an op-amp are always zero. Note that in fig. 75, there is no closed circuit connection for current to flow into or out of the op-amp inputs. This rule will *always hold* for op-amps unless specifically otherwise stated.[106]

[106] This assumption is quite good for modern op-amps which have *very* high resistance looking into the inputs, and you likely won't encounter changes to this rule unless you take a class like EE105.

Golden Rule #2, *Equal Input Voltages (in NFB)*: For an op-amp *in NFB*, $V_+ = V_-$. This rule is quite tricky to understand conceptually, because it seems at first to imply that op-amps in negative feedback always have exactly 0V at the output. Isn't that what it means for the two input voltages to be equal? But remember that in negative feedback, one of the "inputs" is actually some portion of the output! So this rule suggests that the behavior of an op-amp in NFB isn't as simple as it was with the comparator. We had derived that $V_{out} = V_{in}\frac{A}{A+1}$, but note that we can generalize this result for any situation where a portion of the output is sampled and fed into the inverting input. For the unity-gain buffer, this portion was 1 because the output voltage was perfectly (fully) mirrored to the negative input.[107] But if it's some more general fraction f (for feedback factor), then $V_- = f \cdot V_{out}$. The relationship between input and output voltage becomes $V_{out} = V_{in}\frac{A}{Af+1}$ ($\approx \frac{1}{f}$ for large or infinite A). We then apply the previous equation and conclude that $V_- = f \cdot \frac{1}{f}V_{in}$, which is the same as saying $V_- = V_+$ in NFB since $V_{in} = V_+$.

[107] For something like the non-inverting amplifier as we'll see below, this fraction is less than one because of the voltage divider from V_{out} to V_-.

Important Op-Amp Examples: Equipped with knowledge about feedback and op-amp behavior, we can now take a look at 2 very common op-amp configurations

to understand them better. In analyzing circuits, we cannot forget the basics; KCL and NVA still prove very useful in formulating equations with which the circuit can be solved.

Non-inverting Op-Amp: This configuration is shown in fig. 80. The goal is to understand its behavior; specifically, we want to know how V_{out} compares to V_{in} after being amplified.

We must first check whether the golden rules can be applied; #1 certainly can since we have an op-amp, but before using #2, we must check NFB. If V_{out} increases a bit, then the node potential at u_1 increases a bit. To see this, we can recognize that R_1 and R_2 form a voltage divider between V_{out} and ground, so applying the voltage divider equation,[108] eq. (36), $u_1 = V_{\text{out}} \frac{R_2}{R_1+R_2}$. Here, if V_{out} increases, so too does u_1. We then notice that $V_- = u_1$ and so if $V_{\text{out}} \uparrow$, then $V_- \uparrow$ and $A(V_+ - V_-) \downarrow$ so $V_{\text{out}} \downarrow$. The system can correct itself! We've verified NFB.

To solve our system, since the op-amp is in NFB, we can apply Golden Rule #2: $V_- = V_+ \implies u_1 = V_{\text{in}}$.[109] This might be an unexpected result! This configuration enforces that the node potential at u_1 is equal to the input signal voltage!

From here, we apply the voltage divider equation again and perform some algebra:

$$u_1 = V_{\text{out}} \frac{R_2}{R_2 + R_1}$$

$$V_{\text{in}} = V_{\text{out}} \frac{R_2}{R_2 + R_1}$$

$$V_{\text{out}} = V_{\text{in}} \frac{R_2 + R_1}{R_2}$$

We conclude that, for a non-inverting op-amp:

$$V_{\text{out}} = V_{\text{in}} \left(1 + \frac{R_1}{R_2} \right) \tag{48}$$

So based on the specific resistor values used, the non-inverting op-amp allows us to apply a gain to our input signal voltage! For example, if $R_1 = 2\text{M}\Omega$ and $R_2 = 1\text{k}\Omega$ then the gain is $1 + \frac{2 \cdot 10^6}{10^3} = 2001 \approx 2000$. That's pretty good!

Inverting Op-Amp: The previous example showed us how to apply a positive gain, but sometimes we want to turn a negative input signal into a positive output voltage, or vice versa; to do so, we use a configuration as shown in fig. 81. Let's analyze this circuit and solve for its gain, $\frac{V_{\text{out}}}{V_{\text{in}}}$.
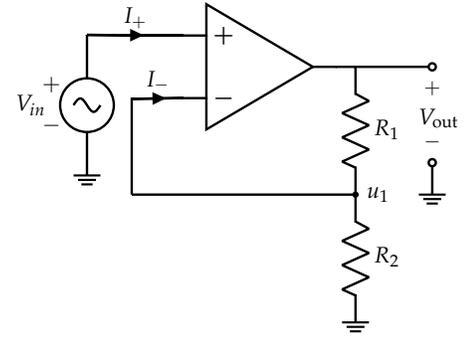
Figure 80: A non-inverting op-amp. Note that the input voltage with the squiggly refers to a *signal voltage*, which is still an independent source but typically of much smaller magnitude (hence the usage of this op-amp to amplify it!)

[108] NVA will yield the exact same result; just be sure to note that the current *coming out of an op-amp* might not be zero; it's only the input currents that are always zero.

[109] If you're tempted to plug in this result to the general op-amp equation, $A(V_{\text{in}} - u_1)$, it's an understandable impulse, but this will only lead you to conclude that $V_{\text{out}} = \infty \cdot 0$, which is undefined and incorrect. One of the hardest things to learn in this unit is to learn how to work towards an expression of the form $\frac{V_{\text{out}}}{V_{\text{in}}}$; this results in a proper expression for the circuit's gain.
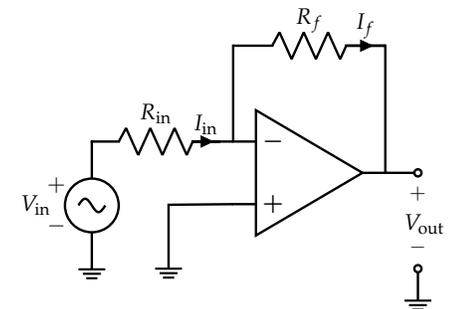
Figure 81: An inverting op-amp. Note that here, there is a so-called "flying" resistor that connects the negative input to the output, creating the NFB connection.

We can begin with a KCL equation at the V_- node. Since we know that op-amps do not allow current into the inputs, $I_- = 0A$. This also means that $I_{\text{in}} = I_f$. We can try applying Golden Rule #2, but let's confirm NFB first as we've done before; if $V_{\text{out}} \uparrow$, then $V_- \uparrow$ (since in the instant the output voltage changes, the current I_f will not change). Then, $A(V_+ - V_-) \downarrow$ so $V_{\text{out}} \downarrow$, and we have NFB. Therefore, we apply Golden Rule #2, and the potential at V_- must be the same as V_+, which is grounded at 0V. Now, combining our findings, this means the same current I_{in} that leads to a V_{in} voltage drop across the R_{in} resistor creates a voltage drop across the R_f resistor, and this second voltage drop directly gives V_{out}! We apply Ohm's law: $V_{\text{in}} = I_{\text{in}} R_{\text{in}} \implies I_{\text{in}} = \frac{V_{\text{in}}}{R_{\text{in}}}$. Since $I_f = I_{\text{in}}$, $0 - V_{\text{out}} = R_f \cdot I_{\text{in}} = R_f \cdot \frac{V_{\text{in}}}{R_{\text{in}}}$.[110] Rearranging this, we find that for an inverting op-amp:

$$V_{\text{out}} = V_{\text{in}} \left(-\frac{R_f}{R_{\text{in}}} \right) \implies \frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{R_f}{R_{\text{in}}} \tag{49}$$

And so we have the gain! While it can be useful to recognize these configurations and immediately jump to the corresponding gain expressions, be sure to also practice systematically analyzing circuits; even encountering differently-drawn versions of these configurations can throw someone off if they've only memorized the results![111] The course notes go into more detail on these and also analyze more complex op-amp configurations.

Loading Effects, Buffering: Let's clarify what exactly loading is in the context of circuits, and present a scenario that will motivate this next section.

Suppose you have a lightbulb, modeled as a simple resistor of 50Ω, and we want to turn on this lightbulb. The greater the power dissipation of the bulb, the brighter it burns, so we want to maximize the voltage dropped across the bulb. One idea we might have is to connect a battery to it, resembling the diagram in fig. 18 where the lightbulb is resistor R. Seems fine, right? Well, the issue is that any battery also has some internal resistance associated with it, and the wires connecting to the bulb have resistance too. Let's model this collective "parasitic" resistance as r, and incorporate it into our diagram so it looks like fig. 82. Now, we have a voltage divider. Let's take a closer look at what happens.

If r is on the order of a couple of ohms, then the voltage divider equation eq. (36) readily reveals that the voltage drop across the lightbulb is the majority of the battery's voltage, so the bulb will burn brightly as desired. But suppose that

[110] Don't forget passive sign convention! The negative terminal is at higher potential relative to V_{out}, and since $V_- = 0$, we can intuitively see why V_{out} is negative.

[111] The semester after I took EECS 16A, I actually confused the inverting and non-inverting op-amp configurations in an interview, and it was the very first question I was asked! I had to recover by performing NVA on the spot; nothing wrong with that!
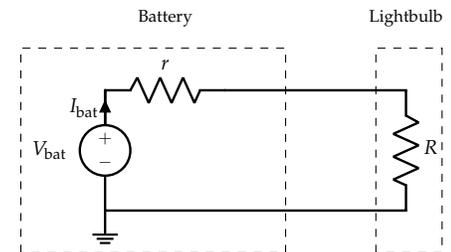
Figure 82: Circuit model for a battery with internal resistance r connected to a lightbulb (which acts as the load here) of resistance R.

we're using a battery with r of 1kΩ. Now, the bulb's voltage is only a small fraction $\frac{50}{1050} \approx 4.7\%$ of the battery's voltage! We'd observe that the bulb's brightness is quite low in this second situation. The issue with this setup arises from the fact that the voltage drop across the bulb depends on the value of r, which is determined by the battery's internal structure. Current flows in this circuit, so some of the battery's voltage is lost to the internal resistance of the battery.[112] Wouldn't it be nice to ensure that no matter what r is, the voltage dropped across the bulb was a constant, *isolating it* from the source? This is exactly what a unity-gain buffer (fig. 78) can be used for.

Let's add this buffer into our circuit, isolating the load (lightbulb) from the source (battery). Our circuit now looks like fig. 83. Notice that the voltage at the non-inverting input is V_{bat}, since no current can flow into the op-amp, and the same voltage appears at the output because of our unity-gain buffer![113] This means that the full V_{bat} will be dropped across the bulb, and we don't need to worry about any loading effects. No matter what the r or R values are,[114], the circuit's behavior is consistent!

Without the buffer in the middle, the impact of the battery's resistance on the bulb depended on specific resistance values. Even using a different kind of wire, with different resistivity or length, could change our observations! By isolating the source and the load, we managed to ensure that no voltage is dropped across the source resistance r and that the load receives the full intended voltage.

Circuit Design Procedure, Current Source Design, Examples: Once again, the notes are a great way to learn this material; design is one of the most interesting aspects of learning circuits because you go a step beyond analyzing what's given to you. Designing circuits helps you appreciate them more, and though it's not covered in the practice sets, it's still crucial to understand! Not to mention, this is a common area from which exam questions have been drawn in the past.

[112] Most often, this just means the battery heats up over time.

[113] For this to happen, by Ohm's law, current must flow *out* of the op-amp; this is perfectly fine since it's only the inputs that current cannot flow into or out of.

[114] If you have an inner voice or tend to read things aloud, you definitely sounded like a pirate there ;)
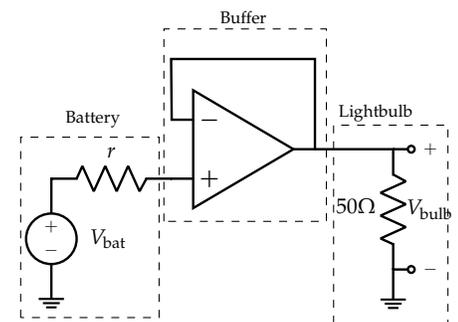
Figure 83: Circuit model for a battery powering a lightbulb, but now with a buffer in the middle. All the voltage with no loss to the battery's internal resistance!

Problems

✠ **Problem:** Is it *true* or *false* that we can use the 2 Golden Rules to analyze any op-amp circuit?

Answer: False

Quick Solution: The first golden rule states that no current can flow into the

inputs of an op-amp, and it has a physical basis; the inputs of any op-amp we'll study are not connected to anything, they're just dangling open circuits. However, the second golden rule, wherein $V_+ = V_-$ does *not* apply to op-amps unless they are wired in negative feedback. Many circuits do not satisfy this criteria, such as when op-amps are used as comparators (see fig. 84 for an example).

✠ **Problem:** Is it *true* or *false* that an op-amp can function without external power supplies?

Answer: False

Quick Solution: Op-amps are limited in their functionality; they simply cannot output a voltage larger or smaller than what is supplied at their rails, and these rail voltages are, by definition, external power supplies. Without these supplies, op-amps will not be able to deliver any power to the load at the output, since the energy must come from somewhere.

✠ **Problem:** Given the circuit in fig. 84, assuming the comparator is *not railing*, find an expression for the output voltage in terms of other variables (gain A, resistor values, supply voltages, and voltage source values).

Answer: $V_{\text{out}} = 2.5 + A(2 - \frac{3}{4}V_{\text{in}})$

Figure 84: Op-amp comparator circuit for the problem to the left. Find an expression for the output voltage!

Quick Solution: Applying the general op-amp output voltage formula for the non-clipped region, eq. (47), the answer follows directly from the rails ($V_{DD} = 5V$, $V_{SS} = 0V$) and the input voltages (voltage divider for V_-, reference source for V_+). Plugging in, $V_{\text{out}} = 2.5V + A\left(2 - \frac{3}{4}V_{\text{in}}\right)$.

Long Solution: The important part of this question is that *despite using an op-amp with very, very high gain as a comparator*, the region of operation is in the non-clipped region (blue line region in fig. 76). This effectively means that we can apply the general op-amp output voltage formula (eq. (47)) to this situation and $V_{\text{out, actual}} = V_{\text{out, ideal}}$. Taking careful note of the supply voltage values and the input voltages, we plug in:

$$V_{\text{out}} = V_{\text{out, ideal}} = \underbrace{\frac{V_{DD} + V_{SS}}{2}}_{\text{rail-determined offset}} + \underbrace{A\left(V_+ - V_-\right)}_{\text{signal gain}}$$

$$V_{\text{out}} = \frac{5V + 0V}{2} + A\left(V_+ - V_-\right)$$

At this point, we observe that the non-inverting input V_+ is being fed directly by an independent voltage source, so $V_+ = 2V$. V_- is a bit trickier, but the key realization is that we form a voltage divider, and V_- is a fraction of V_{in}. Specifically, $V_- = V_{\text{in}} \cdot \frac{3k\Omega}{3k\Omega + 1k\Omega} = \frac{3}{4}V_{\text{in}}$. Substituting in, we have our answer:

$$V_{\text{out}} = 2.5V + A\left(2 - \frac{3}{4}V_{\text{in}}\right)$$

✠ **Problem:** Why does an op-amp not change the voltage of the circuit it is connected to? Phrased another way, why does an op-amp not *load* its input circuitry?

Answer: ∞ input resistance, no input current

Quick Solution: By Golden Rule #1 for all op-amps, the input currents are zero, so when an op-amp is attached to any circuit block, it won't draw additional current, and therefore will not load the circuit block. For an example, consider the voltage divider at V_- of the op-amp in the previous problem. We can apply the voltage divider formula here because adding the op-amp doesn't change the circuit behavior; it doesn't *load* the voltage divider.

Long Solution: By definition, circuit blocks are loaded when they are attached to other blocks/components that change their own behavior. For example, consider the example in the RELEVANT INFORMATION section with the lightbulb and the battery. When the lightbulb is attached, the battery is loaded because the circuit is completed (a battery just sitting there is just a pair of unattached terminals, no closed loop exists.) When current flows in the completed circuit, there is a voltage drop across the internal battery resistance, changing the voltage drop across the lightbulb.

Here, if an op-amp is attached to some circuit block's output, then it cannot draw any current by Golden Rule #1. That is, the op-amp acts as a buffer preventing loading effects. It isolates the "actual load" of the circuit (connected to the output of the op-amp) from the source (at the input of the op-amp), such that the intended load can experience all the voltage benefits as if it was connected to the source, but without loading the source. Op-amps are crucial for allowing circuit blocks to be combined because any time we add a new block to the output of an existing circuit, we don't need to re-analyze the parts of the circuit that came before.

✠ **Problem:** Given some circuit with an op-amp wired in negative feedback, will the gain of the circuit change if the input polarities are swapped (+ becomes − and vice versa)? Why or why not? If it helps, use fig. 80 as an example.

Answer: Yes, negative feedback won't hold.

Quick Solution: Since the circuit was given to be wired in negative feedback before, by swapping the input polarities, we would wire our op-amp in *positive feedback*! Op-amps in positive feedback are highly unstable; minor fluctuations (even from something like thermal noise!) cause the output voltage to become amplified to one of the rails. In NFB, if V_{out} ↑, it self-corrects and soon after, V_{out} ↓. But in positive feedback, if V_{out} ↑$_{small}$, it reinforces and V_{out} ↑↑↑ soon after. Ultimately, this means that Golden Rule #2 no longer applies, and the techniques we used to solve the original circuit and the corresponding gain expression become invalid.

✠ **Problem:** Is it *true* or *false* that an ideal op-amp behaves as though it has infinite

gain?

Answer: True

Quick Solution: This is one of the core features of an ideal op-amp! See eq. (47) graph; the reason the center line is such a narrow region is because of the near-infinite gain. In reality, the gain is very high but not infinite, but the approximation that $V_+ = V_-$ in NFB holds only because we assume $A = \infty$ in this class.

✠ **Problem:** Is the gain of the circuit in fig. 85 positive or negative?

Answer: Negative

Quick Solution: This circuit is identical to fig. 81, which has gain $\frac{V_{\text{out}}}{V_s} = -\frac{R_f}{R_s}$ and so the gain is negative. This is an exercise in pattern-matching and gaining familiarity with recognizing already-seen configurations with slight modifications; the fact that this voltage source is not a signal voltage (that is, it's of somewhat larger magnitude) doesn't change anything about the circuit's fundamental behavior.

Figure 85: Circuit associated with the problem to the left.

✠ **Problem:** Given fig. 85 and the following values, calculate the V_{out} .

$$V_S = 2\text{V}, R_s = 50\Omega, R_f = 100\Omega$$

Answer: $V_{\text{out}} = -4\text{V}$

Quick Solution: We can directly plug into the inverting amplifier gain formula; $V_{\text{out}} = V_s \cdot -\frac{R_f}{R_s} = -2\text{V} \cdot \frac{100}{50} = -4\text{V}$, so $V_{\text{out}} = -4\text{V}$.

Long Solution: Let's review the intuition for why the gain of this configuration is negative, and why the ratio of resistors is the determinant of the value of the gain.

Note that the circuit is in NFB (derived in the RELEVANT INFORMATION section of this chapter). This means Golden Rule #2 applies, and the − input of the op-amp is at 0V (like the + input). The node above the input voltage source is at potential V_s, so by Ohm's law, the current flowing through R_s must contribute V_s voltage drop; that is, $I_s = \frac{V_s}{R_s}$. Now, all op-amps follow Golden Rule #1, so none of this I_s current will flow into the op-amp; all of it goes through R_f. So I_s contributes a voltage drop across the R_f resistor, and by passive sign convention, the left end of R_f is at the higher potential. This left end is at 0V so $V_{\text{out}} < 0\text{V}$, and we can see the gain is negative. If V_s was negative, then current would flow in the other direction and V_{out} would be positive. The ratio $\frac{R_f}{R_s}$ arises from the fact that the same current must go through both resistors, so the ratio of voltages is directly proportional to the ratio of resistances.

✠ **Problem:** Is fig. 86 in negative feedback? .

Answer: Yes

Quick Solution: We can notice that this is a differently-drawn inverting amplifier

Figure 86: Circuit associated with the problem to the left.

(as indicated by the grounded + input and resistors R_1 and R_2 taking the place of R_s and R_f from before. It's quite easy to mistake this for a non-inverting amplifier, based on the way it's drawn; it doesn't change the answer to this specific question, but it's important to be careful!

✠ **Problem:** Solve for V_{out} in fig. 86 in terms of V_{in} and resistor values.

Answer: $V_{\text{out}} = V_{\text{in}} \cdot -\frac{R_2}{R_1}$

Quick Solution: We notice this is an inverting amplifier configuration, and pattern-match accordingly. Performing full NVA or other rigorous analysis will yield the same result. The *very* common mistake is to incorrectly identify the topology of this amplifier and classify it as a non-inverting configuration, leading to an incorrect expression for the output voltage.

✠ **Problem:** Is it *true* or *false* that for reasonable resistor values (positive real resistances), an ideal inverting amplifier (fig. 81) can have any negative gain?

Answer: True

Quick Solution: Looking at the gain expression for an inverting amplifier, $\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{R_2}{R_1}$, it can be seen that with the selection of reasonable resistor values, any negative gain can be achieved (and this was the intended answer). But keep in mind, this is only true since an ideal amplifier has no supply rails; if it did, those would clip the output before it could achieve very negative gains (say, -10^7 as an extreme example).

✠ **Problem:** Is it *true* or *false* that for reasonable resistor values (positive resistances), an ideal non-inverting amplifier (fig. 80) can have any positive gain?

Answer: False

Quick Solution: Unlike the previous question, where an inverting amplifier can apply a gain of, say, $-\frac{1}{10}$ to a signal, a non-inverting amplifier's gain expression is slightly different; as in eq. (48), $V_{\text{out}} = V_{\text{in}} \cdot \left(1 + \frac{R_2}{R_1}\right)$. That is, the ratio of resistances is always added to the value 1, so even in the limit, the minimum gain of a non-inverting amplifier is 1. By itself, it cannot achieve a gain of, say, $\frac{1}{10}$, so it cannot achieve *any* positive gain.[115]

[115] We could do this by chaining together 2 inverting amplifiers though!

✠ **Problem:** Suppose we have 2 circuit blocks A and B, and each one has a specific function (think of each block as having an intended gain or a specific output voltage). We want to chain them together so the output of block A is the input for block B. Will the overall combined circuit's behavior be the same regardless of whether we connect the blocks using a wire (as in fig. 87) or an op-amp (as in fig. 88)?

Figure 87: Two circuit blocks connected by a wire.

Answer: These aren't the same, the op-amp will prevent block B from loading block A unlike the wire.

Figure 88: Two circuit blocks connected by an op-amp.

Quick Solution: Recall the example with the lightbulb and the battery in the RELEVANT INFORMATION section; in this case, we can think of the battery as block A and the bulb as block B. It's the exact same situation that we saw before! Connecting the lightbulb to the battery with just a wire causes the bulb to load the battery (and its internal resistance), so the bulb doesn't experience the full voltage drop. Similarly here, for any arbitrary functional circuit blocks, there's no guarantee that if we add block B to the output of block A using a wire, that there won't be backwards-propagating effects that impact the behavior of block A.

This is where the unity-gain buffer comes in! Attaching block B to block A with a buffer to isolate the two causes the addition of B to have no impact on the intended, designed behavior of A. Since the unity-gain buffer is connected in negative feedback, $V_+ = V_-$ and the output voltage of block A is perfectly mirrored to the input of block B, as desired.

Figure 89: Circuit blocks A (left) and B (right), both are voltage dividers.

Long Solution: *Addendum.* In addition to the QUICK SOLUTION, let's consider a more concrete (and potentially familiar) example. Say blocks A and B are both voltage dividers where for block A, the output voltage is $\frac{1}{4}$ of the input. for block B, the output is $\frac{1}{5}$ of the input. Individually, the circuits are as given in fig. 89.

We can connect these blocks in the style of fig. 87 or fig. 88, with the intention being that we first sample $\frac{1}{4}$ of the input voltage, and then take $\frac{1}{5}$ of that, so the overall end output is $\frac{1}{20}$ of the original input. Does this happen with a wire connection? The combined circuit to analyze is in fig. 90.

If we calculate the block B divider again in this new setup, we find that the voltage division from $V_{\text{out, A}}$ to V_{out} is as expected; $\frac{1}{5}$. But is $\frac{V_{\text{out, A}}}{V_{\text{in}}}$ the ratio of $\frac{1}{4}$ that we would want? If not, then the mere act of attaching block B to the outpu of block A changeed the behavior of block A; by definition, block A has *been loaded by block B*. Let's check!

We can approach this question using resistor network simplifications. Notice that fig. 90 is equivalent to fig. 91. Now, we apply the voltage divider formula and find that since $1\text{k}\Omega \parallel (4\text{k}\Omega + 1\text{k}\Omega) = \frac{5}{6}\text{k}\Omega$, the value of $V_{\text{out, A}} = V_{\text{in}} \cdot \frac{5/6\text{k}\Omega}{3\text{k}\Omega}$, which is $\frac{5}{18} \neq \frac{1}{4}$. Oh no! We've confirmed that the second voltage divider loads the first one, changing its gain from the original $\frac{1}{4}$ to a larger $\frac{5}{18}$. This is why it's necessary to use a buffer to prevent loading in these cases.

It's worth noting that in this case, the amount of loading effect that block B has on block A *is dependent on the raw resistor values* unlike some situations we've seen before (such as the basic voltage divider where the ratio says all). If the resistances in the second divider were on the order of mega-ohms or larger, they'd load block A much less (the parallel simplification would yield a value much closer to 1kΩ.
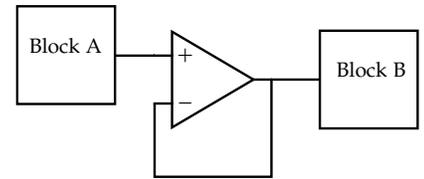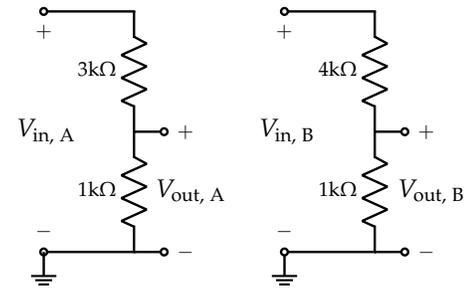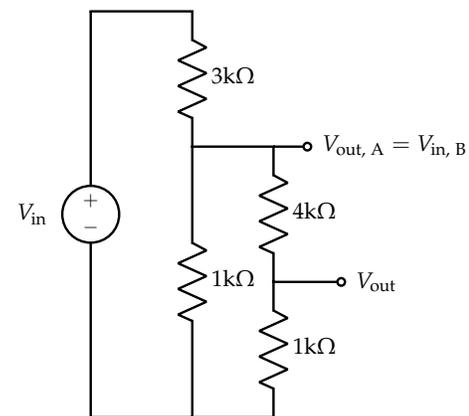
Figure 90: Voltage dividers connected by wire (and a bit of restructuring for clarity), output of A becomes input of B. It should be We want to find how V_{out} relates to V_{in}.
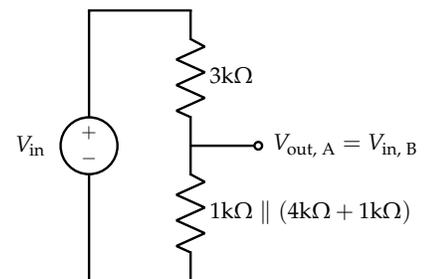
Figure 91: Let's simplify the network of resistors that comes below $V_{\text{out, A}}$ to help analyze the behavior of block A.

Practice Set 10: Locationing and Tri-lateration

Relevant Equations/Information

Brief Motivation: The motivation for this third module is primarily to learn how localization works, developing the linear algebra concepts needed along the way.

Consider how GPS works; there are satellites orbiting the Earth, with known location in space. Given some object, how can we figure out where it is? For each satellite, once we know how far away it is from the device, the distance information from all satellites can be combined to determine a unique position in space.[116]

Questions arise; how can a device figure out the distance to a satellite? What happens if there's no single location that satisfies all the distances measured? How imperfect can these measurements be to still get a reasonably accurate answer? We will answer these questions throughout the remaining chapters.

Norms: First, some terminology. The Euclidean norm (or 2-norm) of a vector $\vec{x} = \begin{bmatrix} x_0 & x_1 & \dots & x_n \end{bmatrix}^\top$ represents the vector's length, or magnitude. It is given by eq. (50); notice we can compute the norm using the efficient inner product operation (defined in eq. (51) below).

$$\|\vec{x}\| = \sqrt{x_0^2 + x_1^2 \dots x_n^2} \equiv \sqrt{\langle \vec{x}, \vec{x} \rangle} \tag{50}$$

Properties:

1. Non-negative Norms: $\|\vec{x}\| \geq 0$, by definition since the magnitude of a vector cannot be negative.

2. Zero Norm: $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$.[117]

3. Scaling a Norm: $\|\alpha \vec{x}\| = \alpha \|\vec{x}\|$.

4. Triangle Inequality: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$. The vectors may have components facing in opposite/different directions, so the magnitude of their vector sum cannot exceed the sum of their magnitudes.[118]

Inner (Dot) Products: Given two vectors $\vec{x}, \vec{y}$ of the same length n, the inner

---

[116] But remember, we need enough satellites! Consider a similar situation where a friend calls you and says that they're 200 feet away from a landmark building (say, Wheeler). Even though you may know the location of Wheeler, your friend could be anywhere around the building (specifically, anywhere on the circle with radius 200 feet, centered at the building). If they specify a second distance to another unique landmark (say, the Golden Bear Cafe), then that would help you narrow your friend's location down!

[117] $\iff$ is the same as "if and only if".

[118] Note the necessary condition: $\vec{x}, \vec{y} \in \mathbb{R}^n$.

product between them can be calculated as:

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \tag{51}$$

For example, if $\vec{x} = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^\top$ and $\vec{y} = \begin{bmatrix} 6 & 0 & 1 & 5 \end{bmatrix}^\top$, then $\langle \vec{x}, \vec{y} \rangle = 0 \cdot 6 + 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 5 = 0 + 0 + 2 + 15 = 17$. Note that the inner product of two vectors yields a *scalar* value, not a vector.

Based solely on the properties of multiplication and addition, we can derive the following about the inner product:

1. inner products are commutative (symmetry): $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

2. scaling either vector by a factor k scales the result by k, and the inner product distributes over vector addition (linearity); $\langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle = \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle$.

3. Positive-Definite (related to the zero-norm property): for any $\vec{u}$, $\langle \vec{u}, \vec{u} \rangle \geq 0$, and $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = 0$.

What significance does an inner product actually have? Generally speaking, it allows us to find the angle between 2 vectors $\vec{x}$ and $\vec{y}$, and this angle provides an approximate sense for the similarity between the vectors. Note that we cannot just look at the value of the inner product $\langle \vec{x}, \vec{y} \rangle$ in making the judgement about the vectors' similarity, due to the scaling property. Making the vectors 10 times longer or shorter doesn't make them any more or less aligned with each other! Rather, we deal with the *normalized* versions of the vectors; we divide each vector by its length so the length of the resulting vector is 1, and we can then focus on its direction alone.

To normalize, we say that $\hat{x} = \frac{\vec{x}}{\|x\|}$. The ^ symbol indicates that we're dealing with a unit vector (magnitude = 1).[119]

Let's derive a generalized result in 2D, with the corresponding diagram in fig. 92. Note that the angle between the vectors is $\theta = \alpha - \beta$ by construction (the vectors themselves have been defined in terms of their angle from the x-axis; the cosine of the angle gives the x-component, the sine gives the y-component). Given any vectors $\vec{x} = c_1 \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ and $\vec{y} = c_2 \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$, we can show that after normalizing, $\hat{x} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ and $\hat{y} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$ since $\sqrt{\cos^2\alpha + \sin^2\alpha} = 1$ and $\|\vec{x}\| = c_1, \|\vec{y}\| = c_2$. Computing:

$$\langle \hat{x}, \hat{y} \rangle = \left\langle \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \right\rangle = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta) = \cos\theta.$$ [120] We
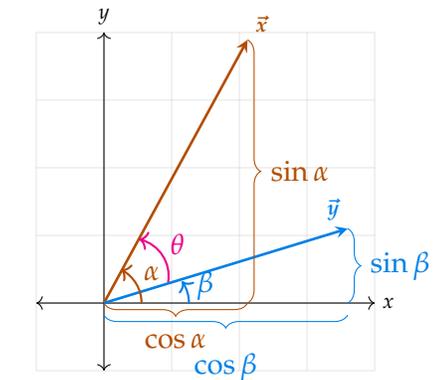
Figure 92: Diagram to help derive how the angle between 2 vectors relates to their inner product.

[119] This is much like $\hat{i}$ or $\hat{j}$, the standard basis vectors in 2D.

[120] This is a trigonometric identity, the difference of cosine angles; see this link for other identities and derivations.

conclude that for unit-magnitude vectors, $\langle \hat{x}, \hat{y} \rangle = \cos\theta$, where θ is the angle between the vectors! Thanks to the scaling property, we can now generalize the equation for vectors of any magnitude:[121]

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \, \|\vec{y}\| \cos\theta. \tag{52}$$

Cauchy-Schwartz Inequality: This theorem states that the magnitude of the inner product of 2 vectors must be less than the product of the norms of the 2 vectors:

$$\left| \langle \vec{x}, \vec{y} \rangle \right| \leq \|\vec{x}\| \, \|\vec{y}\| \tag{53}$$

$\cos\theta$ cannot be larger than 1 for any θ; observing the form of eq. (52), the equation above must hold true. Applying the inner product operation will "preserve" the lengths most when the vectors are perfectly aligned ($\theta = 0°$), but certainly cannot be larger than the individual norms multiplied.

Orthogonal Vectors: Orthogonal is a similar word to perpendicular but is more generally used in higher dimensions,[122] and two vectors are orthogonal if their inner product is zero. For example, the vectors $\vec{x_o}, \vec{y_o}$ in fig. 93 are orthogonal; we see this visually, and we can show this by computing that $\langle \vec{x}, \vec{y} \rangle = 2 \cdot 3 + 3 \cdot -2 = 0$. But the vectors $\vec{x_{no}}, \vec{y_{no}}$ in fig. 94 are not orthogonal, as $\langle \vec{x_{no}}, \vec{y_{no}} \rangle = 1 \cdot 2 + 3 \cdot -3 = -7 \neq 0$.

Vector Operations: The act of computing an inner product is very simple computationally; it's just a few additions and multiplications, so computers are highly optimized for computing inner products. Therefore, it is useful to represent other common operations in terms of inner products. Suppose we define a vector $\vec{x} = \begin{bmatrix} x_0 & x_1 & \ldots & x_n \end{bmatrix}^\top$; a couple simple examples of such operations on $\vec{x}$ follow:

1. Addition: to add the elements in $\vec{x}$, define $\vec{y} \, (\in \mathbb{R}^N) = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^\top$ (the "all ones" vector) and compute $\langle \vec{x}, \vec{y} \rangle = x_0 + x_1 + \ldots + x_n$.
2. Averaging: to average the elements in $\vec{x}$, define the "averaging" vector $\vec{y} \, (\in \mathbb{R}^N) = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} \end{bmatrix}^\top$ and compute $\langle \vec{x}, \vec{y} \rangle = \frac{x_0 + x_1 + \ldots + x_n}{n}$.

2D Intro to Projections: The vector projection of $\vec{x}$ onto $\vec{y}$ refers to the component of $\vec{x}$ that is aligned in the *same direction* as $\vec{y}$ (or exactly opposite; it helps to imagine a line going through $\vec{y}$). Therefore, if the projection of $\vec{x}$ onto $\vec{y}$, denoted proj$_{\vec{y}}\vec{x}$, is zero, then the vectors are orthogonal; there is no component of $\vec{x}$ that is aligned with $\vec{y}$.

Observe fig. 95; here, $\vec{x}$ can be broken up into 2 components; the first ($\vec{x_\parallel}$) is parallel to $\vec{y}$, the second ($\vec{x_\perp}$) is perpendicular. A note on terminology, the *length* of the parallel component $\left\|\vec{x_\parallel}\right\|$ gives the *scalar* projection of $\vec{x}$ onto $\vec{y}$. But we want the

[121] It might be difficult to visualize this intuition for higher dimensions, but we can always define a 2D plane that contains the 2 vectors that we want to take the inner product of (a plane is uniquely defined by 2 lines). Then, when we compute the angle between the vectors, we are finding this angle as measured in that plane.
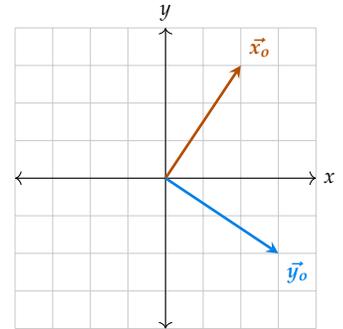
Figure 93: Orthogonal vectors in $\mathbb{R}^2$.
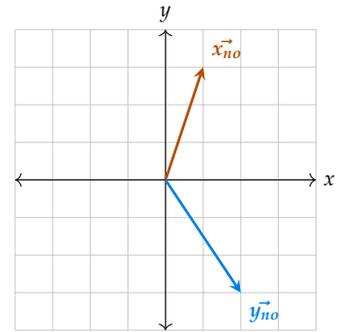
Figure 94: Non-orthogonal vectors in $\mathbb{R}^2$.

[122] See this post for more terminology detail.

vector projection, which is the vector with the same length pointing in the direction of $\vec{y}$. This is what we'll calculate!

Given $\vec{x}$ and $\vec{y}$, how do we calculate the vector projection $\text{proj}_{\vec{y}}\vec{x}$? We note that the length of the parallel component, $\left\|\vec{x}_{\parallel}\right\|$ can be expressed as $\cos\theta \cdot \|\vec{x}\|$, where θ is the angle between $\vec{x}$ and $\vec{y}$. Rearranging the inner product formula (eq. (52)), we find that $\cos\theta = \frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{x}\|\|\vec{y}\|}$. Combining the results: $\left\|\vec{x}_{\parallel}\right\| = \left(\frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{x}\|\|\vec{y}\|}\right)\|\vec{x}\|$. Simplifying:

$$\text{comp}_{\vec{y}}\vec{x} = \left\|\vec{x}_{\parallel}\right\| = \frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{y}\|} \tag{54}$$

This expression is the scalar projection of $\vec{x}$ onto $\vec{y}$! Now that we have this, to find the vector projection, we multiply this scalar by the unit vector in the direction of $\vec{y}$. Since $\hat{y} = \frac{\vec{y}}{\|\vec{y}\|}$, we write that:

$$\text{proj}_{\vec{y}}\vec{x} = \left(\text{comp}_{\vec{y}}\vec{x}\right)\hat{y} \equiv \left(\frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{y}\|}\right)\frac{\vec{y}}{\|\vec{y}\|} \equiv \frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\vec{y} \tag{55}$$

The vector projection comes up most in applications, but it's good to know both kinds! Projections extend beyond 2 dimensions as well; this aspect is discussed more in the next chapter in connection to Least Squares.

Cross-Correlation: For a device to know its location, one of the most important tasks it must do is find the distance to various satellites (which have known positions in space, but unknown locations relative to the device.) The satellites will send known messages, also known as signals; the device must analyze these signals to figure out how far away it's coming from.

What form do the signals take? They are *discrete-time signals* composed of a sequence of values (data points), and these signals are defined only at specific intervals in time (every pair of samples has Δ seconds between them. For convenience I often use $\Delta = 1$.) This is in contrast to a continuous-time signal, which has a defined value at all points in time. See fig. 96 for an example of a discrete-time signal $\left(\vec{s} = \begin{bmatrix} 0 & 1 & -3 & 4 & 1 & -5 & 3 \end{bmatrix}^{\top}\right)$. This signal can be represented either as a graph as in the figure, or as a vector where the value at each index is one of the samples. We will use indexing notation to refer to specific elements of this vector; $s[0]$ is the value (height of the "lollipop") at $t = 0$. For example, $s[0] = 0$ and $s[2] = -3$. Also, these signals have a default value of zero at any indices that aren't contained in the given vector/graph.

To achieve our goal of analyzing a signal to determine the distance to its sender,
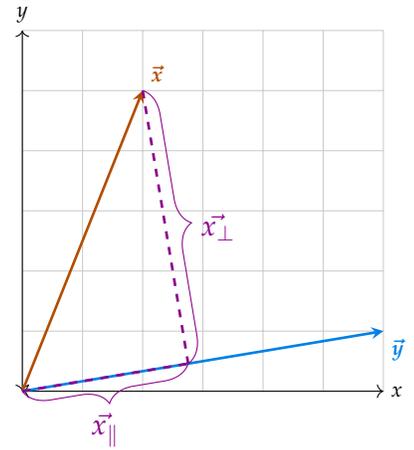
Figure 95: Diagram to illustrate the projection of $\vec{x}$ onto $\vec{y}$.
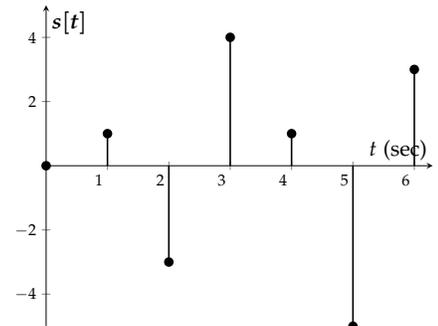
Figure 96: An example of a discrete time signal. Notice that at a time like $0.4s$, there is no defined data value; it only exists at every second.

we need a way of measuring the similarity between 2 signals. The first signal is the message we know the satellite is sending; the second signal is the message the device actually receives, which is the same sequence of values but after some unknown time-delay τ (since the message takes time to get from the satellite to the device). There may also be some *attenuation*; the signal gets weaker as it goes farther, so the received values might be scaled versions of the originals. Once we have the delay τ, then we can find the distance to that satellite as $d = \tau \cdot v$ (velocity of light or sound)

Formally, we write the cross-correlation vector for signals $\vec{x}$, $\vec{y}$ as follows, where $\vec{y}^{(d)}$ denotes a shift (delay) of d seconds:

$$\text{corr}_{\vec{x}}(\vec{y})[d] = \sum_{n=-\infty}^{\infty} x[n]\, y[n-d] = \left\langle \vec{x}, \vec{y}^{(d)} \right\rangle \tag{56}$$

$\vec{x}$ and $\vec{y}$ are the received/sent signals (varies by application) and $\text{corr}_{\vec{x}}(\vec{y})[d]$ is the d-th element of their cross-correlation (at a delay of d seconds).[123] The infinite sum becomes a finite sum since the signals contain zeros everywhere they aren't explicitly defined, leading to zeroed values in the correlation vector. We omit these from the result, focusing on the nonzero values where the signals actually overlap.

Circular Correlation For Locating: Suppose a satellite emits a *periodic* signal $\vec{s_1}$ in the form of a "ramp," as displayed in fig. 97, and the device receives the signal $\vec{r_1}$ in fig. 98. By how much was the signal delayed?[124] We can visually compare $\vec{s_1}$ and $\vec{r_1}$ to determine that the delay is 2 seconds (data points on the $\vec{r_1}$ appear 2 seconds after their corresponding position in the sent signal). The received signal was attenuated by a factor of $\frac{1}{3}$, but the "shape" stayed the same, so we could effectively find the delay.

But what is the computational analog to this comparing process? Given 2 signals, computing and analyzing their cross-correlation can help us figure out the shift (in seconds) the message has experienced in traveling from satellite to device, and this can be used to find the distance! The cross-correlation of two signals is itself a vector, where each index d in the vector represents the inner product between the raw received signal ($\vec{r}[n]$ for all n) and an d-delayed version of the sent signal ($\vec{s}[n-d]$ for all n). As we've seen, the inner product provides a sense for the similarity between 2 vectors, so the higher the value, the more similar the vectors are. The cross-correlation checks the inner product at all relative shifts between the signals, so it tells us how similar two signals are at every shift. Once we have the cross-correlation vector, we
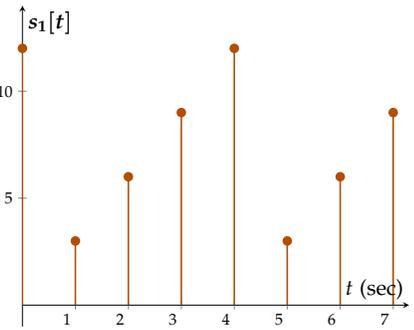
Figure 97: Sample periodic signal that a satellite might send.

[123] If the signals aren't of the same length, no problem! Just add zeros to the smaller vector's end so their lengths match.
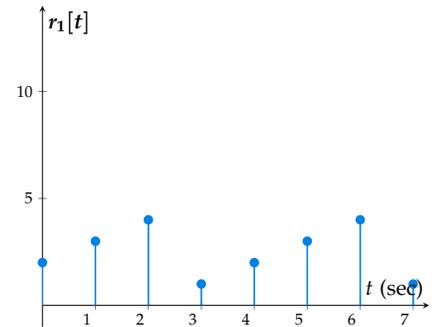
Figure 98: Sample signal that a phone might receive from the satellite in fig. 97.

[124] Note that in practical applications, this delay needs to be positive. You can't receive a message before it's sent!

want to find the index at which the value is maximized, because this index represents the amount of shift/delay the sent signal likely experienced.[125]

So, say we have signals $\vec{s_2}$ and $\vec{r_2}$, and they represent a situation with some delay τ and attenuation factor α. We write that $r_2[d] = \alpha s_2[d - \tau]$. But we can't plot them to visually determine the value of τ or α, and we'd find that trying to solve for these using standard algebraic techniques reveals a nonlinear relationship between τ, $\vec{r}$ and $\vec{s}$. Let's use cross-correlation to find the optimal index.

$$\vec{s_2} = \begin{bmatrix} 6 & 3 & 4 & 1 & 9 & 10 \end{bmatrix}^\top \quad \vec{r_2} = \begin{bmatrix} 2 & 0.5 & 4.5 & 5 & 3 & 1.5 \end{bmatrix}^\top$$

We start by calculating the cross-correlation at zero-shift:

$$\text{corr}_{\vec{r}}(\vec{s})[0] = \sum_{n=-\infty}^{\infty} r[n]\, s[n] = \left\langle \vec{r}, \vec{s}^{(0)} \right\rangle = 12 + 1.5 + 18 + 5 + 27 + 15 = 78.5$$

Next, at a shift of $+1$:

$$\text{corr}_{\vec{r}}(\vec{s})[1] = \sum_{n=-\infty}^{\infty} r[n]\, s[n-1] = \left\langle \vec{r}, \vec{s}^{(1)} \right\rangle = 20 + 3 + 13.5 + 20 + 3 + 13.5 = 73$$

Repeating for all samples, we get the following final vector:

$$\text{corr}_{\vec{r}}(\vec{s}) = \begin{bmatrix} \ldots & 78.5 & 73 & 78.5 & 96.5 & 121.5 & 96.5 & \ldots \end{bmatrix}^\top$$

where the dots represent periodic repetitions. We can conclude that the shift is 4 samples since the 4th index has the largest magnitude (121.5).[126] As a sanity check, this makes sense when viewing the cross-correlation plot, in fig. 99. After finding τ, finding the attenuation is as simple as taking the ratio between corresponding elements in the received signal and the shifted sent signal.

We can find the vector above directly using the following formula for *circular-correlation* (out of scope in Su20), where the N subscript denotes modulo arithmetic:

$$\text{circcorr}(\vec{x}, \vec{y})[k] = \sum_{i=0}^{N-1} x[i]\, y[(i-k)_N] \tag{57}$$

If we expand this out fully, then given $\vec{s}$ and $\vec{r}$, we could write:[127]

$$\text{circcorr}(\vec{r}, \vec{s}) = \begin{bmatrix} - & \vec{s}^{(0)\top} & - \\ - & \vec{s}^{(1)\top} & - \\ & \vdots & \\ - & \vec{s}^{(N-1)\top} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{r} \\ | \end{bmatrix}$$

[125] The cross-correlation of 2 *periodic* vectors will be infinitely long! However, it will itself have the same period as the original signal, so we can focus on just a segment of the cross-correlation vector. Another note; one might wonder how to tell whether the delay in the examples of fig. 98 and fig. 97 was 2 seconds, or actually 6 seconds; they'd look the same at the receiver end! In reality, the satellites might be equipped with clocks to provide a sort of meta-reference.
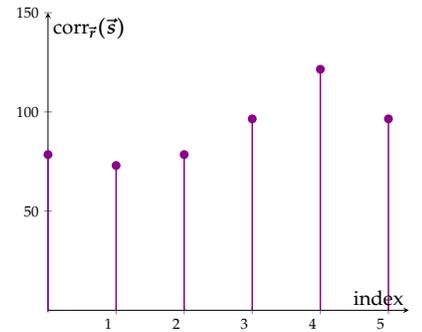
Figure 99: Result of cross-correlation of $\vec{r_2}$ and $\vec{s_2}$.

[126] or $10, 16, 22, \ldots$ samples.

[127] Each row of the circulant matrix represents one of the possible shifts of the signal vector. It's inner products en-masse once we do the matrix-vector multiplication!

$$
= \begin{bmatrix}
s[0] & s[1] & \dots & s[N-2] & s[N-1] \\
s[N-1] & s[0] & \dots & s[N-3] & s[N-2] \\
\vdots & & & & \vdots \\
s[2] & s[3] & \dots & s[0] & s[1] \\
s[1] & s[2] & \dots & s[N-1] & s[0]
\end{bmatrix}
\underbrace{}_{\mathbf{C}_{\vec{s}}}
\begin{bmatrix} | \\ \vec{r} \\ | \end{bmatrix}
$$

$$
= \mathbf{C}_{\vec{s}} \vec{r}
$$

The bracketed matrix is called the *circulant matrix*, and simply populating it using all the shifts of $\vec{s}$ gives a convenient way to use matrix-vector multiplication and jump to the result. This isn't in scope this semester (Su20),[128] and circular correlation is explained more in the notes.[129]

Trilateration: Trilateration is the act of using the distances from the device to each satellite in order to determine device's location. A simple intuitive example to keep in mind is the example with the friend wandering around Berkeley's campus, presented earlier. The sections above cover the details of how to find these distances, but what can we do with them?

We label the scenario as in fig. 100, with a phone somewhere in space with location $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, and satellites at unique locations $\vec{s_1}$, $\vec{s_2}$, $\vec{s_3}$ (of the form $\begin{bmatrix} s_{i,1} \\ s_{i,2} \end{bmatrix}$ for each satellite i). We assume that the satellites are *not* all collinear (their locations do not all lie on the same line); if they did, then we might not be able to uniquely determine a position for the phone.[130] The locations of these are known, as are the distances d_1, d_2, and d_3 from each satellite to $\vec{p}$; the goal is to determine $\vec{p}$.

Applying the concept of norms to vectors, we write the following 3 equations:

1. $\|\vec{p} - \vec{s_1}\|^2 = d_1^2 \implies \vec{p}^\top \vec{p} - 2\vec{s_1}^\top \vec{p} + \|\vec{s_1}\|^2 = d_1^2$

2. $\|\vec{p} - \vec{s_2}\|^2 = d_2^2 \implies \vec{p}^\top \vec{p} - 2\vec{s_2}^\top \vec{p} + \|\vec{s_2}\|^2 = d_2^2$

3. $\|\vec{p} - \vec{s_3}\|^2 = d_3^2 \implies \vec{p}^\top \vec{p} - 2\vec{s_3}^\top \vec{p} + \|\vec{s_3}\|^2 = d_3^2$

While we might hope that this is all we need to solve for the 2D coordinates of $\vec{p}$, it turns out that these $\vec{p}^\top \vec{p}$ terms are problematic; we can't use linear algebra on nonlinear equations! There's a trick we can apply to "sacrifice" one of our equations for the good of the remaining ones; by subtracting the first equation from the second and the third, the second and third equations become linear since the $\vec{p}^\top \vec{p}$ terms

[128] I include it because of the practice problems that ask about it.

[129] Note that both the cross-correlation and circular correlation are not commutative; this intuitively follows from how saying that the sent signal must be shifted forward to match the received signal is not the same as saying that the received signal must be shifted forward to match the sent signal.
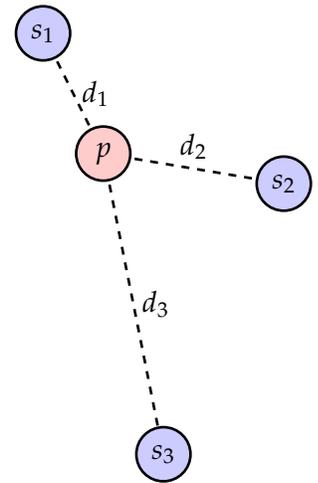
Figure 100: Diagram illustrating 2D locationing of a phone using 3 satellites.

[130] If this is confusing, there's a practice problem about this concept, with some visual examples to illustrate the point in more detail.

disappear:

$$\left(2(\vec{s_1} - \vec{s_2})^\top\right)\vec{p} = \|\vec{s_1}\|^2 - \|\vec{s_2}\|^2 - d_1^2 + d_2^2$$

$$\left(2(\vec{s_1} - \vec{s_3})^\top\right)\vec{p} = \|\vec{s_1}\|^2 - \|\vec{s_3}\|^2 - d_1^2 + d_3^2$$

We can solve the following matrix for $\vec{p}$ since all the matrix entries come from known values:

$$\begin{bmatrix} 2(\vec{s_1} - \vec{s_2})^\top \\ 2(\vec{s_1} - \vec{s_3})^\top \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \|\vec{s_1}\|^2 - \|\vec{s_2}\|^2 - d_1^2 + d_2^2 \\ \|\vec{s_1}\|^2 - \|\vec{s_3}\|^2 - d_1^2 + d_3^2 \end{bmatrix}$$

These 2 linear equations form a solvable linear system in $\vec{p}$ since $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ is a 2-coordinate location! Therefore, assuming ideal conditions (no satellite collinearity) if we have 3 satellites, we can solve for a 2D location; with 5 satellites, we could solve for a 4-coordinate location. In all cases, one of the "dimensions" is always lost since we have to use one equation to linearize the remaining equations. Then, we can apply linear algebra and convert the information into a solvable system.

Problems

✠ **Problem:** Based on the contents of this chapter, given 2 vectors $\vec{v_1}, \vec{v_2} \in \mathbb{R}^N$, what is one way to check whether or not they are orthogonal?

Answer: `assert:` $\langle \vec{v_1}, \vec{v_2} \rangle = 0$, else not orthogonal

Quick Solution: We can take the inner product of the 2 vectors, and the resulting value is related to the cosine of the angle between the vectors (θ) as given by eq. (52). For the vectors to be orthogonal, θ must be 90°; note that we define θ to be measured in the 2D plane containing the 2 vectors of interest, so it doesn't matter how long the vectors themselves are; the logic always holds!

If the result is zero, then we know that $\cos\theta = 0$ (since the vectors themselves are assumed to be nonzero, and their magnitudes must then be nonzero based on a property we derived above). Therefore, $\vec{v_1} \perp \vec{v_2}$ iff $\langle \vec{v_1}, \vec{v_2} \rangle = 0$.

✠ **Problem:** Is it *true* or *false* that for orthogonal vectors $\vec{u}$ and $\vec{v}$ that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$?

Answer: False

Quick Solution: We can generate a simple counter example; the standard basis vectors in 2D. $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, each with magnitude 1. $\hat{i} + \hat{j} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with magnitude $\sqrt{2} \neq 2$.

Alternatively, it is noted in the RELEVANT INFORMATION section that norms

obey the triangle inequality; the extreme case of the sum of norms equaling the norm of sums only happens when the vectors point in the exact *same* direction, not if they're orthogonal to each other. In this second case, the length of the second vector is not in the same direction as the first, so some portion of the second vector's length goes into changing the "direction" of the resulting vector, as opposed to its magnitude.

✠ **Problem:** Is it *true* or *false* that for linearly dependent vectors $\vec{u}$ and $\vec{v}$ that $\|\langle \vec{u}, \vec{v} \rangle\| = \|\vec{u}\|\,\|\vec{v}\|$?

Answer: True

Quick Solution: If the vectors are linearly dependent, then they must be aligned in the same (or exact opposite) directions; this means that the angle between them must be $\theta = 0°$.[131] Observing eq. (52), if $\theta = 0°$, $\cos\theta = 1$, and so the given statement holds directly!

[131] Technically, $\theta = 0$ or ± 180.

Alternatively we can take the inner product for vectors that are linearly dependent ($\vec{v} = \alpha\vec{u}$) as follows::[132]

[132] Note that fluency with the norm/inner product properties is necessary to take this approach.

$$
\begin{aligned}
\|\langle \vec{u}, \vec{v} \rangle\| &= \|\langle \vec{u}, \alpha\vec{u} \rangle\| \\
&= \|\alpha\,\langle \vec{u}, \vec{u} \rangle\| \\
&= \alpha\,\|\langle \vec{u}, \vec{u} \rangle\| \\
&= \alpha\,\|\vec{u}\|^2 \\
&= \alpha\,\|\vec{u}\|\,\|\vec{u}\| \\
&= \|\vec{u}\|\,\|\alpha\vec{u}\| \\
&= \|\vec{u}\|\,\|\vec{v}\| \qquad \blacksquare
\end{aligned}
$$

✠ **Problem:** Suppose we redefine the inner product definition (for 2-element vectors) as follows; previously, $\langle \vec{x}, \vec{y} \rangle = \vec{x}^\top \vec{y}$. But now, $\langle \vec{x}, \vec{y} \rangle = \vec{x}^\top \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{y}$. Does this new definition satisfy the inner product properties (commutativity, linearity, positive-definiteness)? See RELEVANT INFORMATION for a recap of these properties.

Answer: No

Quick Solution: Observing the the middle matrix will effectively "zero out" the second component of $\vec{x}$, we can generate a counterexample to the Positive-Definite property. If $\vec{x} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, then $\langle \vec{x}, \vec{x} \rangle = 0$ even though $\vec{x} \neq 0$. The middle matrix zeroed out an entry that would have been retained in the original definition, and so this new definition doesn't follow the requisite properties.

Long Solution: Suppose that we didn't notice how the matrix zeroed out certain entries, and therefore weren't led to come up a counterexample to that specific property. In that case, we fall back to the rigorous approach apply the formula directly to see if everything's satisfied:

1. Commutativity:

$$\langle \vec{x}, \vec{y} \rangle \overset{?}{=} \langle \vec{y}, \vec{x} \rangle$$

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle \overset{?}{=} \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 y_1 \overset{\checkmark}{=} y_1 x_1$$

2. Linearity:

$$\langle \alpha \vec{x} + \beta \vec{y}, \vec{z} \rangle \overset{?}{=} \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle$$

$$\left\langle \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} + \begin{bmatrix} \beta y_1 \\ \beta y_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle \overset{?}{=} \alpha \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle + \beta \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle$$

$$\begin{bmatrix} \alpha x_1 + \beta y_1 & \alpha x_2 + \beta y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \overset{?}{=} \alpha \left(\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) + \beta \left(\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right)$$

$$(\alpha x_1 + \beta y_1) z_1 \overset{\checkmark}{=} \alpha x_1 z_1 + \beta y_1 z_1$$

3. Positive-Definite (for non-zero $\vec{x}$):

$$\langle \vec{x}, \vec{x} \rangle \overset{?}{\neq} 0$$

$$\left\langle \begin{bmatrix} x_1 \\ \alpha x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ \beta x_2 \end{bmatrix} \right\rangle \overset{?}{=} 0$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \overset{?}{=} 0$$

$$x_1^2 \overset{\checkmark}{=} 0$$

While we've shown that $x_1 = 0$, Notice that there's no restriction on the value of x_2! This second value can be anything; no matter what, it becomes zeroed out by the middle sifting matrix, so the Positive-Definite property is not satisfied; the entire vector $\vec{x}$ does not need to be 0's in order for $\langle \vec{x}, \vec{x} \rangle = 0$.

✠ **Problem:** For $\vec{x}$ and $\vec{y}$, given that $\|\vec{x}\| = 1$, what is $\text{proj}_{\vec{x}} \vec{y}$?

Answer: $\langle \vec{y}, \vec{x} \rangle \, \vec{x}$

Quick Solution: We can use the vector projection formula eq. (55) as a starting point, noting that we're asked here for the projection of $\vec{y}$ onto $\vec{x}$, not $\vec{x}$ onto $\vec{y}$ as in previous examples/derivations:

$$\text{proj}_{\vec{x}} \vec{y} = \left(\frac{\langle \vec{y}, \vec{x} \rangle}{\|\vec{x}\|} \right) \frac{\vec{x}}{\|\vec{x}\|}$$

$$= \left(\frac{\langle \vec{y}, \vec{x} \rangle}{1} \right) \frac{\vec{x}}{1}$$

$$= \langle \vec{y}, \vec{x} \rangle \, \vec{x}$$

✠ **Problem:** Find the vector projection of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ onto $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Answer: $\text{proj}_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 54/25 \\ 72/25 \end{bmatrix}$

Quick Solution: We apply the formula in eq. (55):

$$\text{proj}_{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{\left\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\|^2} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \frac{2 \cdot 3 + 3 \cdot 4}{(\sqrt{3^2 + 4^2})^2} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \frac{18}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 54/25 \\ 72/25 \end{bmatrix}$$

Figure 101: The information from each satellite combines to yield a single, unique location at their intersection.

✠ **Problem:** Is it *true* or *false* that we can always uniquely determine our position in an n dimensional space using $n + 1$ satellites?

Answer: False

Long Solution: The trivial counterexample would be if we have multiple satellites occupying the exact same position in space, so that those satellites are the same distance from the device *and* their equations are the same; then, the redundant satellites aren't providing any new information. For example, if we have 3 satellites on top of each other in 2D space, the device could be anywhere in the circle (that defines the set of points at a certain distance away from the satellite). But what if we say that all satellites need to be at unique locations?

We can try generating counterexamples based on our experience with 2D locations and 3 satellites. In the diagram in fig. 100, we could effectively figure out our 2D position using 3 satellites; one of the distance equations was used to linearize the others, and then the resulting system of linear equations could be solved for a position. The analogous diagram with circle-intersections is in fig. 101. Notice that only 1 location satisfies the constraints from the satellites. Each circle represents the set of possible points for the device based on the information *from that satellite alone*. [133]

However, consider the situation where there are 3 collinear satellites in 2D space; the intersections of the circles would not necessarily be in a single spot. It *could* be, such as in the case of 3 internally tangent circles as in fig. 102, but the resulting trilateration diagram could also look like fig. 103. In this second case,
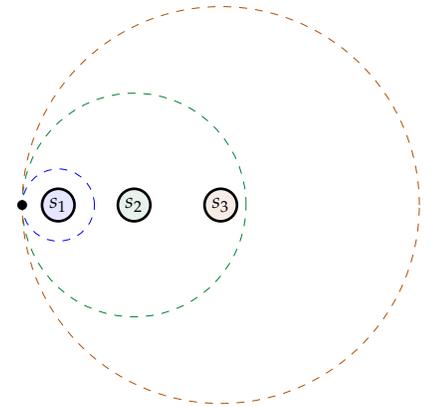
Figure 102: In this special case, collinear satellites are perfectly located so that they can uniquely define a device position.
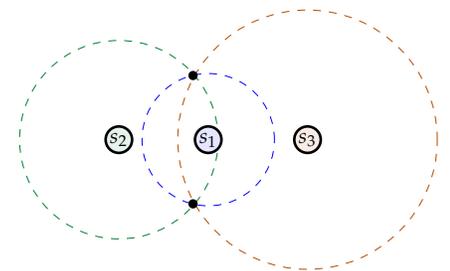
Figure 103: Since the satellites are collinear, it is possible to find a configuration such as this one, where there is not just one potential location for the device.

[133] In 3D, we would have spheres instead of circles, and hyperspheres in higher dimensions.

there are 2 distinct points that both satisfy the distance measurements of all 3 satellites, so we cannot *always* uniquely determine the device's position.

✠ **Problem:** As a device that knows a satellite's periodic message $\vec{s}$, you receive a vector $\vec{r}$. Calculating $\text{circcorr}(\vec{r}, \vec{s}) = \mathbf{C}_{\vec{s}}\vec{r}$, you find the vector is as given below. How many samples of delay would you estimate the signal underwent in traveling to you from the satellite?

$$\text{circcorr}(\vec{r}, \vec{s}) = \begin{bmatrix} -32 & -32 & 16 & 16 & 96 & 24 \end{bmatrix}^{\top}$$

Answer: 4 samples

Quick Solution: The cross-correlation vector gives the inner product between the received message $\vec{r}$ and every version of the shifted sent signal ($\vec{s}^{(0)}, \vec{s}^{(1)}, \ldots$). The index in the vector gives the amount of shift (for example, the inner product between $\vec{r}$ and $\vec{s}^{(0)}$ (no shift) yielded -32). Since the inner product is a measure of the similarity between 2 vectors, the index of the cross-correlation vector where the value is the highest denotes the shift applied to the sent signal for maximum similarity. In this case, since the 4th index (96) is the highest value in the vector, we conclude that the sent signal underwent 4 samples of delay in travelling to the receiver. We had to shift the signal 4 samples forward in order to best match the received signal.

✠ **Problem:** Is it *true* or *false* that $\|\mathbf{C}_{\vec{x}}\vec{y}\| = \|\mathbf{C}_{\vec{y}}\vec{x}\|$?

Answer: True

Quick Solution: Note that this question concerns the *norms*; not the circular correlations themselves. Observing the formula for circular correlation in eq. (57), the only difference between the 2 expressions $\text{circcorr}(\vec{x}, \vec{y})$ and $\text{circcorr}(\vec{y}, \vec{x})$ is the index in which a given inner product will appear; the entries themselves will be the same, just reversed. Therefore, the norms will be the same, since the calculation for a vector's norm depends only on the values themselves, not their indices within the vector.

✠ **Problem:** Is it *true* or *false* that the result of projecting $\vec{y}$ onto $\vec{v}$ is the same as the projection of $\vec{y}$ onto a nonzero, scaled vector $\alpha\vec{v}$?

Answer: True

Quick Solution: Note that the length of the vector being *projected onto* doesn't factor into the value of the vector projection *at all* in the end; the only aspect of the vector being projected onto that affects the vector projection is its direction, since we scale it down to a unit vector before multiplying it by the scalar projection.

Long Solution: Consider eq. (55); scaling $\vec{y}$ by a constant multiplies the inner product by the constant, the norm by the square of the constant, and the vector itself again by the constant; the combined effects cancel out!

Using the properties of norms and inner products, we can show this rigorously; projecting onto a linearly dependent, nonzero version of a vector is the same as the projection onto the original vector.

$$\text{proj}_{\vec{v}}\,\vec{y} \overset{?}{=} \text{proj}_{\alpha\vec{v}}\,\vec{y}$$

$$\left(\frac{\langle \vec{y}, \vec{v}\rangle}{\|\vec{v}\|}\right) \frac{\vec{v}}{\|\vec{v}\|} \overset{?}{=} \left(\frac{\langle \vec{y}, \alpha\vec{v}\rangle}{\|\alpha\vec{v}\|}\right) \frac{\alpha\vec{v}}{\|\alpha\vec{v}\|}$$

$$\left(\frac{\langle \vec{y}, \vec{v}\rangle}{\|\vec{v}\|}\right) \frac{\vec{v}}{\|\vec{v}\|} \overset{?}{=} \left(\frac{\alpha\,\langle \vec{y}, \vec{v}\rangle}{\alpha\,\|\vec{v}\|}\right) \frac{\alpha\vec{v}}{\alpha\,\|\vec{v}\|}$$

$$\left(\frac{\langle \vec{y}, \vec{v}\rangle}{\|\vec{v}\|}\right) \frac{\vec{v}}{\|\vec{v}\|} \overset{\checkmark}{=} \left(\frac{\langle \vec{y}, \vec{v}\rangle}{\|\vec{v}\|}\right) \frac{\vec{v}}{\|\vec{v}\|}$$

The constant α terms cancelled out, as expected!

Practice Set 11: Least Squares

Relevant Equations/Information

Why Least Squares?: In Module 1, we primarily concerned ourselves with techniques to figure out if the system at hand had a unique solution. If there was no set of values that could satisfy the system, we concluded that it had no solutions. But in reality, almost all of the systems we solve will have noisy data and measurements, so arriving at a unique solution is quite rare. In the absence of a solution vector that satisfies all the given information, what can we do?

The least squares technique yields a solution that satisfies all the given equations/information *as closely as possible* (based on our definition of "close," presented below). By collecting more information (so we have more equations than unknowns), we will focus on solving over-determined systems. These may not (likely *will* not) have a single set of variable values that satisfy all the equations, but least squares will allow us to reject the measurement noisiness as much as possible.

What is Least Squares?: The least squares problem can be formulated as follows: Find an optimal "solution"[134] vector $\vec{x}$ such that the difference between the actual observed values ($\vec{b}$) and the system's *predicted* values ($\underset{m \times n}{\mathbf{A}} \vec{x}$) is as small as possible. We denote this difference the "error vector" $\vec{e}$, the magnitude (norm) of which we seek to minimize. The term "least squares" arises from the fact that $\|\vec{e}\| = \sqrt{e_1^2 + e_2^2 + \cdots + e_n^2}$. We are trying to minimize the sum of the squares of $\vec{e}$'s elements.

[134] Note that solution is in quotes since $\vec{x}$ will not satisfy many of the equations individually; the goal is to make it match the system as a whole as best as possible.

Mathematically, we have that $\vec{e} = \vec{b} - \mathbf{A}\vec{x}$, or that $\vec{b} - \vec{e} = \mathbf{A}\vec{x}$; recalling the matrix-vector product (columns-based interpretation) from many chapters ago, eq. (8), we recognize that $\mathbf{A}\vec{x}$ is a linear combination of the columns of $\mathbf{A}$[135], with weights given by the elements of $\vec{x}$. We want to minimize the error, which means selecting weights $x_1, \ldots, x_n$ to pick a point in col(**A**) as close to $\vec{b}$ as possible.

[135] Note that the span of the column space of **A** is denoted col(**A**), as in Module 1 chapters.

Based on the dimensions of **A** alone (more equations that unknowns, $m > n$), we can see that the columns span an n-dimensional subspace within the larger m-dimensional space that $\vec{e}$ and $\vec{b}$ lie in, so $\vec{b}$ cannot be exactly reached by $\mathbf{A}\vec{x}$.

Generalized Minimization of Error $\vec{e}$: In the previous chapter, we derived the 2D

scalar and vector projection formulae; here, we introduce least squares notation and connect it to the idea of minimizing the distance between $\vec{b}$ and col(**A**).

Consider fig. 104, a very similar diagram to fig. 95 but labeled differently to ease the transition to least squares analysis. We decided that the projection of $\vec{b}$ onto $\vec{a_1}$ should be the parallel component of $\vec{b}$ that aligns in the same direction as $\vec{a_1}$, and this would be some scaled portion $\alpha\vec{a_1}$. Any other scalar $\alpha' \neq \alpha$ would increase the error vector's magnitude as $\vec{e}$ would no longer be orthogonal to $\vec{a_1}$.[136]

We can generalize this logic to higher dimensions; choosing weights $\vec{x}$ for the columns of **A** creates a vector, and the set of all these vectors for all $\vec{x}$ defines the column space of **A**. Given that $\vec{b}$ contains some known information, our task is to select the specific $\vec{x}$ where the error is orthogonal to col(**A**), because intuitively, this will minimize $\|\vec{e}\| \equiv \left\|\vec{b} - \mathbf{A}\vec{x}\right\|$. That is, for a system defined by the corresponding matrix **A**, we want to find $\vec{x}$ where $\vec{e} \perp$ col(**A**).[137]

Deriving Least Squares Formula: We've established that an optimal vector $\vec{e}$ will satisfy that $\vec{e} \perp$ col(**A**), meaning $\vec{e} \perp \vec{a_i}$ for each column of **A**.[138]

Mathematically, for all $i \in 0\ldots n$:

$$\langle \vec{e}, \vec{a_i} \rangle = 0$$
$$\implies \left\langle \vec{b} - \mathbf{A}\vec{x}, \vec{a_i} \right\rangle = 0$$
$$\implies \vec{a_i}^\top \left(\vec{b} - \mathbf{A}\vec{x} \right) = 0$$

The last equation follows from how the inner product can be viewed as a matrix multiplication of a row vector and a column vector. Writing out these equations:

$$\vec{a_1}^\top \left(\vec{b} - \mathbf{A}\vec{x} \right) = 0$$
$$\vec{a_2}^\top \left(\vec{b} - \mathbf{A}\vec{x} \right) = 0$$
$$\vdots$$
$$\vec{a_n}^\top \left(\vec{b} - \mathbf{A}\vec{x} \right) = 0$$

The transpose $\vec{a_i}^\top$ of a column vector $\vec{a_i}$ is a row vector, we can vertically combine all these equations (the $(\vec{b} - \mathbf{A}\vec{x})$ term is the same for all of them, and the 0 becomes a vector, $\vec{0}$):

$$\begin{bmatrix} - & \vec{a_1}^\top & - \\ - & \vec{a_2}^\top & - \\ & \vdots & \\ - & \vec{a_n}^\top & - \end{bmatrix} \left(\vec{b} - \mathbf{A}\vec{x} \right) = \vec{0}$$

Figure 104: 2D vector projection, with least squares notation.

[136] If this doesn't make intuitive sense, the notes derive why this is the case.

[137] There are two approaches to more rigorously proving this result; geometrically (for 3D at least) and algebraically (for any dimension, using the properties of inner products). Both are shown in the notes.

[138] Linear combinations of these column vectors are in col(**A**) by definition, so the individual vectors must be in col(**A**) also.

Now, we've formed the transpose of $\mathbf{A}$ on the left side! So:

$$\mathbf{A}^\top \left(\vec{b} - \mathbf{A}\vec{x}\right) = \vec{0}$$

Solving for $\vec{x}$ in terms of the known quantities (entries of $\mathbf{A}$ and $\vec{b}$ come from the collected data/information), being sure to follow the basic rules of matrix multiplication and inversion:

$$\mathbf{A}^\top \left(\vec{b} - \mathbf{A}\vec{x}\right) = \vec{0}$$
$$\mathbf{A}^\top\vec{b} - \mathbf{A}^\top\mathbf{A}\vec{x} = \vec{0}$$
$$\mathbf{A}^\top\mathbf{A}\vec{x} = \mathbf{A}^\top\vec{b}$$
$$\vec{x} = \left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top\vec{b} \qquad (58)$$

And there we have it! The final least squares equation is given by eq. (58). We made an important assumption in the derivation above, namely that $\mathbf{A}^\top\mathbf{A}$ is invertible. Is this a valid assumption? As it turns out,[139] if $\mathbf{A}$ has independent columns, then $\mathbf{A}$ has a trivial null-space (as does the square matrix $\underset{n\times n}{\mathbf{A}^\top\mathbf{A}}$). $\mathbf{A}^\top\mathbf{A}$ is then invertible!

[139] See the course notes for details!

The final concept to consider is how the predicted values of $\vec{b}$ (we'll call this $\vec{b}_{\text{pred}}$) can be calculated from our calculated optimal $\vec{x}$, and how this compared to the original $\vec{b}$ values. Since we have an expression for $\vec{x}$, we find that:

$$\vec{b}_{\text{pred}} = \mathbf{A}\vec{x} = \mathbf{A}\left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top\vec{b}$$

And the error, by definition (and as expected), is $\vec{e} = \vec{b} - \vec{b}_{\text{pred}}$.

Basic Worked Least Squares Example:

Example: Find the best-fit line for the points $\{(1,0),(1.5,3),(2,3.5),(5,6)\}$. See fig. 105.

Figure 105: Plotting the given points for the example problem.

Solution: Since no line (expressed as $y = mx + b$) goes through all 4 points, this is not a system with an exact solution. We also have 4 points (4 pieces of information, 4 equations), but only 2 points are needed to define a line; since the system has more equations than unknowns (constants m and b in the equation for a line), we can apply least squares here to reject the impact of noise. For this function, the inputs are the x-coordinates, and the outputs are the y-coordinates.

Using the given data, we want to construct a model that can take other inputs and give predictions of the output. Note the assumption here that the underlying situation or model that generated these data *is actually linear*; we're trying to model the system by solving for the slope m and intercept b. It's just that the data is noisy

so the points don't actually fall on a line as they should. Feel free to guess at the approximate equation for a best-fit line, and see how the results compare!

The equation given by the first point, $(1, 0)$, is $m \cdot 1 + b \equiv m + b = 0$. The second point gives $1.5m + b = 3$, and so on. Formulating our equations as a matrix-vector product of coefficients (using the points we have) and unknowns (m, b):

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1.5 & 1 \\ 2 & 1 \\ 5 & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 3 \\ 3.5 \\ 6 \end{bmatrix}$$

We can apply the least squares equation in eq. (58), noting that all dimensions match as they should for the multiplications to be valid:

$$\vec{x} = \left(\mathbf{A}^\top \mathbf{A} \right)^{-1} \mathbf{A}^\top \vec{b}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \left(\begin{bmatrix} 1 & 1.5 & 2 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1.5 & 1 \\ 2 & 1 \\ 5 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1.5 & 2 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 3.5 \\ 6 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 32.25 & 9.5 \\ 9.5 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 41.5 \\ 12.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.103226 & -0.245161 \\ -0.245161 & 0.832258 \end{bmatrix} \begin{bmatrix} 41.5 \\ 12.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1.21912 \\ 0.229044 \end{bmatrix}$$

Figure 106: The data from $\vec{b}$ is plotted in blue, the predictions from $\vec{b}_{\text{pred}}$ are in black. The error for each measurement (called thee *residual*) is in red. Naturally, since the predictions came from the best-fit line we obtained via least squares analysis, they coincide.

Based on this result, our best-fit line has slope $m = 1.21912$ and y-intercept $b = 0.229044$; that is $y = 1.21912x + 0.229044$.[140] How good is this result? Let's superimpose the line with the raw data in fig. 106. We can also calculate

[140] The numbers aren't that nice; data in practical applications rarely is :'(

$$\vec{b}_{\text{pred}} = \mathbf{A}\vec{x} = \begin{bmatrix} 1 & 1 \\ 1.5 & 1 \\ 2 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1.21912 \\ 0.229044 \end{bmatrix} = \begin{bmatrix} 1.44816 \\ 2.05772 \\ 2.66728 \\ 6.32464 \end{bmatrix}$$

Plotting these points in black in fig. 106, we can see how they compare to the data. Looks like the line is a good fit!

The (optimal) error vector is $\vec{e} = \begin{bmatrix} -1.44816 \\ 0.94228 \\ 0.83272 \\ -0.32464 \end{bmatrix}$, with negative entries indicating

where the model overestimates the data. $\|\vec{e}\| = 1.9452$; no other line can yield a smaller error. For example, say we picked the similar equation $y = 1.25x + 0.25$; $\|\vec{e}\|$ would be $1.956 > 1.9452$.

Problems

✠ **Problem:** Is it *true* or *false* that least squares is a method used to solve under-determined systems of linear equations?

Answer: False

Quick Solution: Least squares is actually used in the cases of *over-determined* systems. The reason that we want to collect as much information and data as possible about a system, even though this makes it very unlikely that we can arrive at a unique solution (using a technique such as gaussian elimination) is so that we can arrive at as good a solution as possible. This solution is one that minimizes the error between the observed and predicted values.

The equations probably cannot be all simultaneously solved by a single $\vec{x}$, but using least squares will allow us to reject the data's noise as much as possible and model the underlying linear system as accurately as possible.

✠ **Problem:** Given the equation below, solve for the least squares optimal $\vec{x}$.

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ -8 \end{bmatrix}$$

Answer: $\vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Quick Solution: We can directly apply the least squares formula we've derived, eq. (58):

$$\vec{x} = \left(\mathbf{A}^\top \mathbf{A} \right)^{-1} \mathbf{A}^\top \vec{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -8 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

✠ **Problem:** Is it *true* or *false* that least squares always has a unique solution (that is, for $\vec{x} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \vec{b}$, $\vec{x}$ is unique)?

Answer: False

Quick Solution: If the column vectors of $\mathbf{A}$ are not linearly independent, then null($\mathbf{A}$) (and therefore null($\mathbf{A}^\top \mathbf{A}$)) are not trivial. If $\mathbf{A}^\top \mathbf{A}$ has a nontrivial null-space, it is not invertible, so a unique solution to the system cannot be found.

Let's consider why we would have multiple solutions (infinite, actually) in the case of nontrivial null($\mathbf{A}^\top \mathbf{A}$). Note that by definition of a nontrivial null-space, there exists a linear combination of the columns of null($\mathbf{A}^\top \mathbf{A}$) that equals zero. You can then always add such a linear combination to whatever optimal $\vec{x}$ solution we obtain, yielding yet another $\vec{x}'$ that minimizes $\|\vec{e}\|$. We can always remove the redundant columns from $\mathbf{A}$ to ensure that least squares yields a unique solution.

✠ **Problem:** Is it *true* or *false* that least squares can be applied to solve for coefficients a, b in the equation $y = a \sin(bx)$ given pairs of points on the line?

Answer: False

Quick Solution: Note that this equation is not linear *on a and b*, and therein lies the problem; the existence of sin itself is not an issue since we could perfectly take pairs of points on an equation like $y = a \sin(x) + b$, and use these to solve for an optimal a and b. But, the moment we encounter a situation where the equation is nonlinear on a, b, the problem can no longer be tackled with least squares.

Therefore, since both a and b are describing the sine wave, creating a nonlinear relationship, we would not be able to construct a linear least squares setup for this problem given what we know.

✠ **Problem:** Assume the following scenario: we have some data (height, weight, age, white blood cell count) for each patient in a group. We want to use this sample data to generate a function of the form $\mathbf{A}\vec{x} = \vec{b}$ in order to predict white blood cell counts for a larger group of patients. What information will populate $\mathbf{A}$?

Answer: height, weight, and age

Quick Solution: This situation follows the patterns for the scenarios we've seen before; if the predictor variable is the white blood cell count, then our goal is to find out how to weigh the other information (height, weight, age) in order to

best approximate the blood cell count. These variables (which we can treat as the inputs to our function) form the columns of $\mathbf{A}$, while $\vec{x}$ contains the associated weights and $\vec{b}$ contains the predicted variable, the blood cell counts. The function itself could be a very complicated function of the inputs, but it must be linear in the unknowns (weights, x_i) to be solvable using least squares.[141]

✠ **Problem:** Is it *true* or *false* that if $\vec{x} = \text{proj}_{\text{col}(\mathbf{A})}\vec{b}$ is the projection of a vector $\vec{b}$ onto the column space of $\mathbf{A}$, that $\mathbf{A}^\top\left(\vec{b} - \vec{x}\right) = \vec{0}$?

Answer: True

Quick Solution: We know that $\vec{x}$ lies in $\text{col}(\mathbf{A})$, whereas $\vec{b}$ doesn't necessarily. This means that (just as we did in the vector projection derivation in fig. 95 in the previous chapter and fig. 104 in this chapter) we can isolate the component of $\vec{b}$ that is orthogonal to $\text{col}(\mathbf{A})$ as $\vec{b} - \vec{x}$. Note that we use perpendicular projections, so if this leftover component (call it $\vec{b}_\perp$) is orthogonal to $\text{col}(\mathbf{A})$, it is also orthogonal to each of the individual column vectors of $\mathbf{A}$ (which each lie in the column space).

Therefore, if we decompose $\mathbf{A}^\top\left(\vec{b} - \vec{x}\right)$ into each row i's equation as $\vec{a}_i^\top\left(\vec{b} - \vec{x}\right)$, the result will be zero for all rows (and the resulting vector is therefore $\vec{0}$).[142] Similar logic was applied in deriving the least squares equation in the RELEVANT INFORMATION section above, but in the other direction (composing $\mathbf{A}^\top$ from rows rather than decomposing it into rows).

Long Solution: *Some other insights*: It is important to recognize that $\vec{b}$ is only in the null-space of $\mathbf{A}^\top$, since it is orthogonal to every *column* in $\mathbf{A}$. $\vec{b}$ does *not* belong to the null-space of $\mathbf{A}$ (these matrices have different null-spaces!) This becomes especially clear when we consider that these matrices have different dimensions, and therefore the dimensions of their null-spaces must be different (we can invoke the Rank-Nullity theorem eq. (16), since $\text{rank}(\mathbf{A}) = \text{rank}((\mathbf{A}^\top))$).

We can also now interpret the null-space in a slightly different way; if $\vec{x}$ is in $\text{null}(\mathbf{A})$ then $\mathbf{A}\vec{x} = 0$ by definition. Also, $\vec{x}$ is orthogonal to each row $\vec{a}_i^\top$ of $\mathbf{A}$. $\text{null}(\mathbf{A})$ therefore is comprised of all vectors that are orthogonal to every row in $\mathbf{A}$ (every column in $\mathbf{A}^\top$).

✠ **Problem:** Is it *true* or *false* that the projection of $\vec{b}$ onto a set of vectors $\{a_1, a_2, \ldots, a_k\}$ $= \frac{\langle \vec{b},\vec{a_1}\rangle}{\langle \vec{a_1},\vec{a_1}\rangle}\vec{a_1} + \frac{\langle \vec{b},\vec{a_2}\rangle}{\langle \vec{a_2},\vec{a_2}\rangle}\vec{a_2} + \cdots + \frac{\langle \vec{b},\vec{a_k}\rangle}{\langle \vec{a_k},\vec{a_k}\rangle}\vec{a_k}$?

Answer: False

Quick Solution: If $\{a_1, a_2, \ldots, a_k\}$ are not orthogonal, then adding the vector projections could lead to double-counting of overlapping components (see example in the LONG SOLUTION). We would end up arriving at an incorrect projection, unless we first found an orthogonal basis for $\text{span}(\{a_1, a_2, \ldots, a_k\}$ and then computed the vector projections onto those orthogonal vectors.

Long Solution: In this case, we have no information about the set of vectors

[141] Not necessary for the problem at all, but why might a hospital want to do this? Well, blood tests are expensive and time-consuming! Measuring height, weight, and age is far easier relatively, so if they can reasonably approximate white blood cell counts for a patient given the easier-to-collect info, that would be quite helpful and efficient.

[142] Note that $\vec{a}_i$ is a column vector in $\mathbf{A}$, so its transpose is a row-vector in $\mathbf{A}^\top$.

$\{a_1, a_2, \ldots, a_k\}$, specifically about their orthogonality. If the vectors happen to all be mutually orthogonal, then the statement would be true, because each individual vector projection term would be in a distinct direction; they can be added to yield the overall projection without "cross-contamination" across directions. Here, there might be overlaps between the terms corresponding to $\vec{a_1}$ and $\vec{a_2}$, for example, leading to an incorrect result. Consider the simple case of the projection of a vector $\vec{b}$ onto the set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$. These are linearly dependent, so the projections onto them are the same; $\text{proj}_{\vec{a_1}} \vec{b} = \text{proj}_{\vec{a_2}} \vec{b}$. [143] Adding the projections here would be double-counting!

[143] If this is unclear, see the previous chapter's RELEVANT INFORMATION section, or the very last problem of the previous chapter for a more focused treatment of this idea.

To solve this issue, we would need to take the set of vectors $\{a_1, a_2, \ldots, a_k\}$ and find an *orthogonal basis* for their span, which can be done using an iterative procedure like Gram-Schmidt orthogonalization (out of scope in Su20). Again, anything out-of-scope or not covered in class notes may not be valid on exams.

✠ **Problem:** Is it *true* or *false* that for an arbitrary cost (error) function, the error vector $\vec{e}$ corresponding to the optimal approximation of $\vec{b}$ onto some col(A) will be orthogonal to col(A)? That is, is $\vec{e} \perp$ col(A) always?

Answer: False

Quick Solution: If we used the standard definition of cost $= \|\vec{e}\|$, then indeed, just as we have experienced with least squares, $\vec{e}$ will be orthogonal to col(A). However, cost functions can be quite complex,[144] and not all of them have the guarantee that $\vec{e} \perp$ col(A). This question didn't specify a cost-function, so we cannot assume anything about how to solve the system.

[144] see LONG SOLUTION for an explanation of cost functions and more context.

Long Solution: Let's take a moment (2 paragraphs) to clarify the concept of a cost function, and how this connects to the terminology we know.

A cost function takes in an input (or several) and outputs a value, and this magnitude of this value represents the degree of error in our approximation. In the case of least squares, the cost function is the norm operation; it takes the error vector as the input and outputs the norm as a number. Why do we need a cost function at all? It's a convenient metric to abstract complex details about the actual error into a single value; otherwise, given error vectors [1, 2], and [0, 3], how can we quantify which is better? What if the error is represented as a non-vector-like object? The cost function takes care of these calculations.

Different applications may warrant different cost functions; for example, suppose that for some nuanced situation, given an error vector $\vec{e}$, having any error value $e_i > 5$ is catastrophic; for any such point we will add e_i^4 rather than e_i^2 the square when computing the modified "norm" of $\vec{e}$. Then, $\vec{e_1} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ evaluates to a cost of $\sqrt{5^2 + 5^2} = 7.07$ than an error vector of $\vec{e_2} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ with cost $\sqrt{6^4 + 0^2} = 36$. The standard norm operation would lean in favor of $\vec{e_2}$ (with cost 6) instead of $\vec{e_2}$ (with cost 7.07). We use cost functions to represent quantities to minimize; there exists another category called reward functions which we want to maximize. The

core concept of optimizing (minimizing or maximizing) a single output value is the same.

Now, for the question: if we used the standard definition of an error vector's norm as our cost function, then as we have seen with least squares, the error $\vec{e}$ will be orthogonal to $\text{col}(\mathbf{A})$. However, consider a cost function of the form $\text{cost}(\vec{e})$ $= |e_1| + |e_2| + \cdots + |e_n|$; this does not have the guarantee of orthogonality. Or, consider a simpler (but perhaps useless?) cost function that always outputs 16; this also will not have such properties.

While it isn't in scope to solve for any arbitrary cost function, it is worth knowing what exactly the least squares process is minimizing, and how to compute a basic cost function's output value given its definition and inputs. The next question is an example of a computation we can perform knowing the specific cost function.

✠ **Problem:** Given the system of equations below, and the cost function $c(x)$ below that, find the optimal approximation $\vec{x}$ to solve the system.

$$\begin{cases} \alpha_1 x = \beta_1 \\ \alpha_2 x = \beta_2 \end{cases}$$

$$c(x) = 2(\beta_1 - \alpha_1 \vec{x})^2 + (\beta_2 - \alpha_2 \vec{x})^2$$

Answer: $\vec{x} = \frac{2\alpha_1\beta_1 + a_2\beta_2}{2\alpha_1^2 + \alpha_2^2}$

Long Solution: This question deals with a concept that we haven't yet seen (in this book); a calculus-based approach to minimize a cost function by differentiating the expression with respect to the variable of interest (here, $\vec{x}$) and setting that equal to 0.

In least squares, the cost function is different than the one given here; we take the norm of the error vector and that gives us the representation of the cost, which gives a measure of how close our model's predictions are to the data. Calculating this error vector requires that we *already have* the optimal vector $\vec{x}$, since $\vec{e} = \vec{b} - \mathbf{A}\vec{x}$. In least squares, we can solve for the optimal $\vec{x}$ using eq. (58). But here, the cost function has been differently defined, and we haven't derived a formula that gives us $\vec{x}$ directly. But, the underlying concept is the same; to find the minimum value of a function, we need to differentiate it and solve for the value of x that makes it zero. So:

$$\begin{aligned} \frac{d}{dx}c(x) &= \frac{d}{dx}\left(2\left(\beta_1^2 - 2\alpha_1\beta_1\vec{x} + \alpha_1^2\vec{x}^2\right) + \left(\beta_2^2 + 2\alpha_2\beta_2\vec{x} + \alpha_2^2\vec{x}^2\right)\right) \\ &= \frac{d}{dx}\left(4\beta_1^2 - 4\alpha_1 b_1\vec{x} + \alpha_1^2\vec{x}^2 + \beta_2^2 + 2\alpha_2\beta_2\vec{x} + \alpha_2^2\vec{x}^2\right) \\ &= \left(4\alpha_1\beta_1 + 2\alpha_1^2\vec{x} + 2\alpha_2\beta_2 + 2\alpha_2^2\vec{x}\right) \\ &= \left(4\alpha_1^2 + 2\alpha_2^2\right)\vec{x} - 4\alpha_1\beta_1 - 2\alpha_2\beta_2 \end{aligned}$$

Setting this $= 0$, solving for $\vec{x}$:

$$0 = \left(4\alpha_1^2 + 2\alpha_2^2\right)\vec{x} - 4\alpha_1\beta_1 - 2\alpha_2\beta_2$$

$$4\alpha_1\beta_1 + 2\alpha_2\beta_2 = \left(4\alpha_1^2 + 2\alpha_2^2\right)\vec{x}$$

$$\frac{4\alpha_1\beta_1 + 2\alpha_2\beta_2}{4\alpha_1^2 + 2\alpha_2^2} = \vec{x}$$

$$\frac{2\alpha_1\beta_1 + \alpha_2\beta_2}{\alpha_1^2 + \alpha_2^2} = \vec{x}$$

And we have our expression for $\vec{x}$ in the last row!