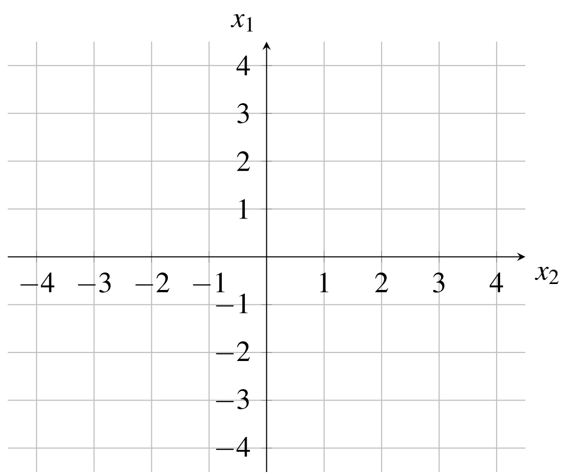
CSM 16A Fall 2020 — Designing Information Devices and Systems I — Week 12

1. Projections

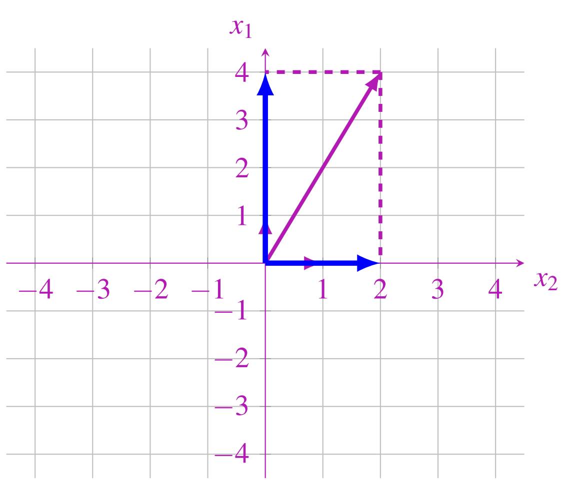
Learning Goal: The goal of this problem is to understand the properties of projection.

Relevant Notes: Note 23 walks through mathematical derivations for projection.



(a) Consider the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Draw it on the graph provided. Also draw the vector $\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, find the projections of $\vec{x}$ on $\vec{y}_1$ and $\vec{y}_2$ geometrically. Compare with mathematical calculations.

Answer:



The projection of vector $\vec{b}$ on vector $\vec{a}$ is given by:

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\left\langle \vec{b}, \vec{a} \right\rangle}{\|\vec{a}\|^2}\vec{a} \tag{1}$$

Now we have:

$$\langle \vec{x}, \vec{y_1} \rangle = 2 \cdot 1 + 4 \cdot 0 = 2$$
$$\langle \vec{x}, \vec{y_2} \rangle = 2 \cdot 0 + 4 \cdot 1 = 4$$
$$\|\vec{y_1}\| = \sqrt{1+0} = 1$$
$$\|\vec{y_2}\| = \sqrt{0+1} = 1$$

Hence projection of vector $\vec{x}$ on vector $\vec{y_1}$ is

$$\text{proj}_{\vec{y_1}}\vec{x} = \frac{\langle \vec{x}, \vec{y_1} \rangle}{\|\vec{y_1}\|^2}\vec{y_1} = \frac{2}{1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \tag{2}$$
$$\text{proj}_{\vec{y_2}}\vec{x} = \frac{\langle \vec{x}, \vec{y_2} \rangle}{\|\vec{y_2}\|^2}\vec{y_2} = \frac{4}{1}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\4\end{bmatrix} \tag{3}$$

(b) Calculate the projection of $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$ on $\vec{y} = \begin{bmatrix}3\\4\end{bmatrix}$. Is it the same as the projection of $\vec{y}$ on $\vec{x}$?

Answer: Hence projection of vector $\vec{x}$ on vector $\vec{y}$ is

$$\text{proj}_{\vec{y}}\vec{x} = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}\vec{y} = \frac{1 \cdot 3 + 1 \cdot 4}{3^2 + 4^2}\begin{bmatrix}3\\4\end{bmatrix} = \frac{7}{25}\begin{bmatrix}3\\4\end{bmatrix}. \tag{4}$$

Now the projection of vector $\vec{y}$ on vector $\vec{x}$ is given by

$$\text{proj}_{\vec{x}}\vec{y} = \frac{\langle \vec{y}, \vec{x} \rangle}{\|\vec{x}\|^2}\vec{x} = \frac{3 \cdot 1 + 4 \cdot 1}{1^2 + 1^2}\begin{bmatrix}1\\1\end{bmatrix} = \frac{7}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3.5\\3.5\end{bmatrix}, \tag{5}$$

which is not the same as $\text{proj}_{\vec{x}}\vec{y}$.

(c) Now consider the vectors $\vec{x} = \begin{bmatrix}1\\2\\4\end{bmatrix}$, $\vec{y_1} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\vec{y_2} = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Now, find the projections of $\vec{x}$ on $\vec{y_1}$ and $\vec{y_2}$. Also find the projection of $\vec{x}$ on $\text{span}\{\vec{y_1}, \vec{y_2}\}$. Is $\text{proj}_{\vec{y_1}}\vec{x} + \text{proj}_{\vec{y_2}}\vec{x}$ equal to $\text{proj}_{\text{span}\{\vec{y_1},\vec{y_2}\}}\vec{x}$? Explain your answer.

Answer: The projection of vector $\vec{x}$ on vector $\vec{y_1}$ is

$$\text{proj}_{\vec{y_1}}\vec{x} = \frac{\langle \vec{x}, \vec{y_1} \rangle}{\|\vec{y_1}\|^2}\vec{y_1} = \frac{1 \cdot 1 + 2 \cdot 1 + +4 \cdot 0}{1^2 + 1^2 + 0^2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1.5\\1.5\\0\end{bmatrix}. \tag{6}$$

Similarly projection of vector $\vec{x}$ on vector $\vec{y_2}$ is

$$\text{proj}_{\vec{y_2}}\vec{x} = \frac{\langle \vec{x}, \vec{y_2} \rangle}{\|\vec{y_2}\|^2}\vec{y_2} = \frac{1 \cdot 0 + 2 \cdot 0 + +4 \cdot 1}{0^2 + 0^2 + 1^2}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{4}{1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\4\end{bmatrix}. \tag{7}$$

We can use the least squares formula to find the projection of a vector on a subspace. The the projection of $\vec{x}$ on span$\{\vec{y}_1, \vec{y}_2\}$ is the same as projection of $\vec{x}$ on col$\{\mathbf{A}\}$, where matrix $\mathbf{A}$ has the columns $\vec{y}_1$ and $\vec{y}_2$.

$$\text{proj}_{\text{span}\{\vec{y}_1,\vec{y}_2\}}\vec{x} = \mathbf{A}\hat{\vec{\alpha}} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{x}, \tag{8}$$

where $\hat{\vec{\alpha}}$ is the least squares solution to $\mathbf{A}\vec{\alpha} = \vec{x}$.

Now we can calculate:

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(\mathbf{A}^T\mathbf{A})^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^T\vec{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

So we can calculate

$$\text{proj}_{\text{span}\{\vec{y}_1,\vec{y}_2\}}\vec{x} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{x} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \\ 4 \end{bmatrix}$$

.

Now summing up the projections on $\vec{y}_1$ and $\vec{y}_2$, we have

$$\text{proj}_{\vec{y}_1}\vec{x} + \text{proj}_{\vec{y}_2}\vec{x} = \begin{bmatrix} 1.5 \\ 1.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \\ 4 \end{bmatrix} = \text{proj}_{\text{span}\{\vec{y}_1,\vec{y}_2\}}$$

. Here $\text{proj}_{\vec{y}_1}\vec{x} + \text{proj}_{\vec{y}_2}\vec{x}$ is equal to $\text{proj}_{\text{span}\{\vec{y}_1,\vec{y}_2\}}\vec{x}$ since $\vec{y}_1$ and $\vec{y}_2$ are orthogonal, i.e. $\langle \vec{y}_1, \vec{y}_2 \rangle = 0$.

(d) Find the expression for projection of $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ on the columnspace of matrix $\mathbf{A} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Is $\text{proj}_{\vec{a}_1}\vec{b} + \text{proj}_{\vec{a}_2}\vec{b}$ equal to $\text{proj}_{\text{Col}\{\mathbf{A}\}}\vec{b}$? (No need to do the calculations.)

If we set up a system of linear equations $\mathbf{A}\vec{x} = \vec{b}$, will there be a unique solution? (No need to solve the system.)

Answer: We use the least squares formula again to find the projection of a vector on a subspace. The the projection of $\vec{b}$ on Col$\{\mathbf{A}\}$ is

$$\text{proj}_{\text{Col}\{\mathbf{A}\}}\vec{b} = \mathbf{A}\hat{\vec{\mathbf{x}}} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}, \tag{9}$$

where $\hat{\vec{\mathbf{x}}}$ is the least squares solution to $\mathbf{A}\vec{\mathbf{x}} = \vec{b}$.

Now we can calculate:

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$(\mathbf{A}^T\mathbf{A})^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$$

$$\mathbf{A}^T\vec{x} = \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\2\\4\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix}$$

So we can calculate

$$\text{proj}_{\text{Col}\{\mathbf{A}\}}\vec{b} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} = \begin{bmatrix}1 & 1\\ 1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}0.5 & -0.5\\ -0.5 & 1.5\end{bmatrix}\begin{bmatrix}7\\1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0.5 & -0.5\\ 0.5 & -0.5\end{bmatrix}\begin{bmatrix}7\\1\end{bmatrix} = \begin{bmatrix}1\\3\\3\end{bmatrix}.$$

The columns of $\mathbf{A}$ are not orthogonal, so $\text{proj}_{\vec{a}_1}\vec{b} + \text{proj}_{\vec{a}_2}\vec{b}$ is not equal to $\text{proj}_{\text{Col}\{\mathbf{A}\}}\vec{b}$.

There won't be a unique solution to $\mathbf{A}\vec{x} = \vec{b}$, since $\text{proj}_{\text{Col}\{\mathbf{A}\}}\vec{b} \neq \vec{b}$. This means that $\vec{b}$ is not in the columnspace of $\mathbf{A}$, so the system is inconsistent.

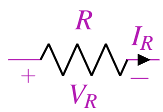
2. And You Thought You Could Ignore Circuits Until Dead Week

Learning Goal: The objective of this problem is to practice solving a noisy system using least squares method.

Relevant Notes: Note 23 covers the details of least squares method.

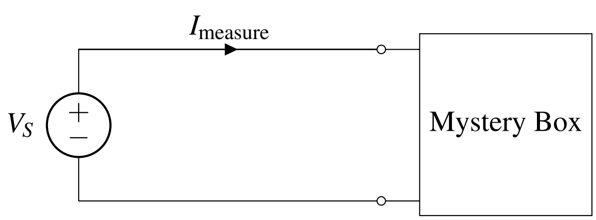
(a) Write Ohm's Law for a resistor.

Answer: For the resistor



$$V_R = I_R R$$

(b) You're given the following test setup and told to find R_{eq} between the two terminals of the mystery box. What is R_{eq} of the mystery box between the two terminals in terms of V_S and I_{measure}?



Answer:

$$R_{eq} = \frac{V_S}{I_{\text{measure}}}$$

$$R_{eq} = \frac{V_S}{I_{\text{measure}}}$$

(c) You think you've figured out how to find R_{eq}! You've taken the following measurements:

Measurement #	V_S	I_{measure}
1	2V	1A
2	4V	2A
3	6V	2A
3	8V	4A

Using the information above, formulate a least squares problem whose answer provides an estimate of R_{eq}.

Answer: According to Ohm's Law, $V_S = I_{measure}R_{eq}$. We have to calculate the least squares solution $\hat{R_{eq}}$. We are estimating the resistance, so $\hat{R}_{eq}$ corresponds to $\hat{\vec{x}}$ in the equation $\mathbf{A}\hat{\vec{x}} = \vec{b}$, where $\vec{i} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ correspond to $\mathbf{A}$ and $\vec{b}$, respectively:

$$\begin{aligned}
\hat{\vec{x}} &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} \\
\hat{R_{eq}} &= (\vec{i}^T\vec{i})^{-1}\vec{i}^T\vec{v} \\
&= (\begin{bmatrix} 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix})^{-1} \begin{bmatrix} 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \\
&= (1 \cdot 1 + 2 \cdot 2 + 2 \cdot 2 + 4 \cdot 4)^{-1}(1 \cdot 2 + 2 \cdot 4 + 2 \cdot 6 + 4 \cdot 8) \\
&= (25)^{-1}(54) \\
&= \frac{54}{25}\Omega = 2.16\Omega
\end{aligned}$$

(d) Find the least square error vector $\|\vec{e}\|$.

Answer: The minimum error is given by

$$\vec{e} = \mathbf{A}\hat{\vec{x}} - \vec{b} = \vec{i}\hat{R}_{eq} - \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix} \times 2.16 - \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2.16 \\ 4.32 \\ 4.32 \\ 8.64 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0.16 \\ 0.32 \\ -1.68 \\ 0.64 \end{bmatrix}$$

3. Least Squares Fitting

Learning Goal: The objective of this problem is to set up a least squares problem for coefficients of non-linear equations.

Relevant Notes: Note 23 covers the details of least squares method.

In an upward career move, you join the starship USS Enterprise as a data scientist. One morning the Chief Science Officer, Mr. Spock, hands you some data for the position (y) of a newly discovered particle at different times (t). The data has three points and **contains some noise**:

$$(t = 0, y = 0.5), \quad (t = 1, y = 3), \quad (t = 2, y = 18.5)$$

Your research shows that the path of the particle is represented by the function:

$$y = e^{w_1 + w_2 t} \tag{10}$$

You decide to fit the collected data to the function in Equation (**??**) using the Least Squares method.

series = qn You need to find the coefficients w_1 and w_2 that *minimize the squared error* between the fitted curve and the collected data points. So you set up a system of linear equations, $\mathbf{A}\hat{\vec{\alpha}} \approx \vec{b}$ in order to find the approximate value of $\hat{\vec{\alpha}} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. What are the values of $\mathbf{A}$ and $\vec{b}$?

Answer: For $t = 0$, we have:

$$\begin{aligned} 0.5 &= e^{w_1 + w_2(0)} \\ \implies \ln(0.5) &= w_1 + 0(w_2) \end{aligned}$$

Similarly for $t = 1$, and $t = 2$, we have:

$$\begin{aligned} \ln(3) &= w_1 + w_2 \\ \ln(18.5) &= w_1 + 2w_2 \end{aligned}$$

Hence we can write the following system of linear equations:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \approx \begin{bmatrix} \ln(0.5) \\ \ln(3) \\ \ln(18.5) \end{bmatrix}$$

seriies = qn Mr. Spock thinks one of the data points is wrong and asks you to redo the fit with only two data points. What will happen to the norm of the error, $\|\vec{e}\| = \left\| \vec{b} - \mathbf{A}\hat{\vec{\alpha}} \right\|$?

Answer: The linear system now has two unknowns (w_1, w_2) and two linearly-independent constraints (the two data points), so there will be an exact fit to the data: the norm of error $\|\vec{e}\| = \left\| \vec{b} - \mathbf{A}\hat{\vec{\alpha}} \right\|$ will be 0. This is probably too good to be true!

seriiies = qn Your colleague tries to repeat your fitting process with the same four data points in part (a), but they misread the equation relating t and y, i.e. they use the following function (which is **different than part (a)**):

$$y = e^{w_1 t + w_2 t} \tag{11}$$

Your colleague tries to find w_1 and w_2 by setting up a system of equations $\mathbf{A}\hat{\vec{\alpha}} \approx \vec{b}$ and utilizing the equation:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \hat{\vec{\alpha}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}. \tag{12}$$

What will happen when your colleague tries to solve the above equation?

Answer: The new system of linear equations can be written as:

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \approx \begin{bmatrix} \ln(0.5) \\ \ln(3) \\ \ln(18.5) \end{bmatrix}$$

Notice that the first and second columns of $\mathbf{A}$ are the same. Since $\mathbf{A}$ has linearly dependent columns, $\mathbf{A}^T\mathbf{A}$ will not be invertible, i.e. the equation for $\hat{\vec{\alpha}}$ will not work.

4. Besto Pesto (Final Exam, Fall 2018) [PRACTICE]

Your TA Laura is struggling to keep her basil plant alive! She needs your help to determine how much water and sunlight her plant needs.

Let $x_h[k]$ be the plant's height on day k and $x_\ell[k]$ be the number of leaves on the plant on day k. The vector $\vec{x}[k] = \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}$ defines the state of the plant. The evolution of the basil plant from one day to the next is defined by the **approximate** mathematical model:

$$\vec{x}[k+1] = \mathbf{A}\vec{x}[k] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}. \tag{13}$$

(a) Our first goal is to estimate the elements of state transition matrix, $\mathbf{A}$: $a_{11}, a_{12}, a_{21}, a_{22}$. To do this we count the leaves and measure the height for the first N time steps, i.e. we know $\{\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[N]\}$.

Setup a least squares problem to estimate $\vec{a} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$:

$$\hat{\vec{a}} = \underset{\vec{a}}{\text{argmin}} ||\mathbf{M}\vec{a} - \vec{b}||^2. \tag{14}$$

Write the matrix, M, and vector, $\vec{b}$, that would be used in the above least squares problem for $N = 3$.

Answer: We can simplify the equations:

$$\begin{aligned} \begin{bmatrix} x_h[k+1] \\ x_\ell[k+1] \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix} \\ \implies x_h[k+1] &= a_{11}x_h[k] + a_{12}x_\ell[k] \\ x_\ell[k+1] &= a_{21}x_h[k] + a_{22}x_\ell[k] \end{aligned}$$

For $N = 3$, we have:

$$\begin{aligned} x_h[1] &= a_{11}x_h[0] + a_{12}x_\ell[0] \\ x_\ell[1] &= a_{21}x_h[0] + a_{22}x_\ell[0] \\ x_h[2] &= a_{11}x_h[1] + a_{12}x_\ell[1] \\ x_\ell[2] &= a_{21}x_h[1] + a_{22}x_\ell[1] \\ x_h[3] &= a_{11}x_h[2] + a_{12}x_\ell[2] \\ x_\ell[3] &= a_{21}x_h[2] + a_{22}x_\ell[2] \end{aligned}$$

$$\mathbf{M} = \begin{bmatrix} x_h[0] & x_\ell[0] & 0 & 0 \\ 0 & 0 & x_h[0] & x_\ell[0] \\ x_h[1] & x_\ell[1] & 0 & 0 \\ 0 & 0 & x_h[1] & x_\ell[1] \\ x_h[2] & x_\ell[2] & 0 & 0 \\ 0 & 0 & x_h[2] & x_\ell[2] \end{bmatrix} \tag{15}$$

$$\vec{b} = \begin{bmatrix} x_h[1] \\ x_\ell[1] \\ x_h[2] \\ x_\ell[2] \\ x_h[3] \\ x_\ell[3] \end{bmatrix} \tag{16}$$