1. Projections

Learning Goal: The goal of this problem is to understand the properties of projection.

Relevant Notes: Note 23 walks through mathematical derivations for projection.

(a) Consider the vector \( \vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \). Draw it on the graph provided. Also draw the vector \( \vec{y}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \vec{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Now, find the projections of \( \vec{x} \) on \( \vec{y}_1 \) and \( \vec{y}_2 \) geometrically. Compare with mathematical calculations.

(b) Calculate the projection of \( \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) on \( \vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \). Is it the same as the projection of \( \vec{y} \) on \( \vec{x} \)?
(c) Now consider the vectors \( \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \) and \( \vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \). Now, find the projections of \( \vec{x} \) on \( \vec{y}_1 \) and \( \vec{y}_2 \). Also find the projection of \( \vec{x} \) on \( \text{span}\{\vec{y}_1, \vec{y}_2\} \). Is \( \text{proj}_{\vec{y}_1} \vec{x} + \text{proj}_{\vec{y}_2} \vec{x} \) equal to \( \text{proj}_{\text{span}\{\vec{y}_1, \vec{y}_2\}} \vec{x} \)? Explain your answer.

(d) Find the expression for projection of \( \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \) on the columnspace of matrix \( \mathbf{A} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \).

Is \( \text{proj}_{\vec{a}_1} \vec{b} + \text{proj}_{\vec{a}_2} \vec{b} \) equal to \( \text{proj}_{\text{Col}(\mathbf{A})} \vec{b} \)? (No need to do the calculations.)

If we set up a system of linear equations \( \mathbf{A} \vec{x} = \vec{b} \), will there be a unique solution? (No need to solve the system.)

2. And You Thought You Could Ignore Circuits Until Dead Week

Learning Goal: The objective of this problem is to practice solving a noisy system using least squares method.

Relevant Notes: Note 23 covers the details of least squares method.

(a) Write Ohm’s Law for a resistor.

(b) You’re given the following test setup and told to find \( R_{eq} \) between the two terminals of the mystery box. What is \( R_{eq} \) of the mystery box between the two terminals in terms of \( V_S \) and \( I_{\text{measure}} \)?
(c) You think you’ve figured out how to find $R_{eq}$! You’ve taken the following measurements:

<table>
<thead>
<tr>
<th>Measurement #</th>
<th>$V_S$</th>
<th>$I_{measure}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2V</td>
<td>1A</td>
</tr>
<tr>
<td>2</td>
<td>4V</td>
<td>2A</td>
</tr>
<tr>
<td>3</td>
<td>6V</td>
<td>2A</td>
</tr>
<tr>
<td>3</td>
<td>8V</td>
<td>4A</td>
</tr>
</tbody>
</table>

Using the information above, formulate a least squares problem whose answer provides an estimate of $R_{eq}$.

(d) Find the least square error vector $\|\vec{e}\|$.

3. Least Squares Fitting

**Learning Goal:** The objective of this problem is to set up a least squares problem for coefficients of non-linear equations.

**Relevant Notes:** Note 23 covers the details of least squares method.
In an upward career move, you join the starship USS Enterprise as a data scientist. One morning the Chief Science Officer, Mr. Spock, hands you some data for the position \( y \) of a newly discovered particle at different times \( t \). The data has three points and contains some noise:

\[
(t = 0, y = 0.5), \quad (t = 1, y = 3), \quad (t = 2, y = 18.5)
\]

Your research shows that the path of the particle is represented by the function:

\[
y = e^{w_1 + w_2 t}
\]  

(1)

You decide to fit the collected data to the function in Equation (1) using the Least Squares method. You need to find the coefficients \( w_1 \) and \( w_2 \) that minimize the squared error between the fitted curve and the collected data points. So you set up a system of linear equations, \( A \hat{\alpha} \approx \bar{b} \) in order to find the approximate value of \( \hat{\alpha} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \). What are the values of \( A \) and \( \bar{b} \)?

Mr. Spock thinks one of the data points is wrong and asks you to redo the fit with only two data points. What will happen to the norm of the error, \( \| \bar{e} \| = \| \bar{b} - A \hat{\alpha} \| \)?

Your colleague tries to repeat your fitting process with the same four data points in part (a), but they misread the equation relating \( t \) and \( y \), i.e. they use the following function (which is different than part (a)):

\[
y = e^{w_1 t + w_2 t}
\]  

(2)

Your colleague tries to find \( w_1 \) and \( w_2 \) by setting up a system of equations \( A \hat{\alpha} \approx \bar{b} \) and utilizing the equation:

\[
\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \hat{\alpha} = (A^T A)^{-1} A^T \bar{b}.
\]  

(3)

What will happen when your colleague tries to solve the above equation?
4. Besto Pesto (Final Exam, Fall 2018) [PRACTICE]

Your TA Laura is struggling to keep her basil plant alive! She needs your help to determine how much water and sunlight her plant needs.

Let \( x_h[k] \) be the plant’s height on day \( k \) and \( x_l[k] \) be the number of leaves on the plant on day \( k \). The vector \( \vec{x}[k] = \begin{bmatrix} x_h[k] \\ x_l[k] \end{bmatrix} \) defines the state of the plant. The evolution of the basil plant from one day to the next is defined by the approximate mathematical model:

\[
\vec{x}[k+1] = A \vec{x}[k] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_l[k] \end{bmatrix}.
\] (4)

(a) Our first goal is to estimate the elements of state transition matrix, \( A \): \( a_{11}, a_{12}, a_{21}, a_{22} \). To do this we count the leaves and measure the height for the first \( N \) time steps, i.e. we know \{\( \vec{x}[0], \vec{x}[1], \ldots, \vec{x}[N] \}\).

Setup a least squares problem to estimate \( \hat{\vec{a}} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} \):

\[
\hat{\vec{a}} = \arg\min_{\vec{a}} \| M \vec{a} - \vec{b} \|^2.
\] (5)

Write the matrix, \( M \), and vector, \( \vec{b} \), that would be used in the above least squares problem for \( N = 3 \).