1. Proof on Linear (In)Dependence [WALK-THROUGH]

**Learning Goal:** The goal of this problem is to practice some proof development skills.

(a) Show that if the system of linear equations, $A \vec{x} = \vec{0}$, has a non-zero solution, then the columns of $A \in \mathbb{R}^{m \times n}$ are linearly dependent.

We are going to use the approach outlined in Note 4. Please also look into Note 3 Subsection 3.1.1 for the definition of linear dependence/independence.

(i) **Start with what we already know:**
   We know that system of equations, $A \vec{x} = \vec{0}$, has a non-zero solution, $\vec{u}$. Express this information in a mathematical form.

(ii) **Then consider what we need to show:**
   We have to show that the columns of $A$ are linearly dependent. Using the definition of linear dependence from Note 3 Subsection 3.1.1, write a mathematical equation that conveys linear dependence of columns of $A$.

(iii) **How to go from “what we know” to “what we need to show”:**
   Now manipulate the expression from (i) using mathematically logical steps to reach the expression from part (ii).

(b) Show that if the system of linear equations: $A \vec{x} = \vec{b}$, has at least one solution for $A \in \mathbb{R}^{m \times n}$, then $\vec{b}$ should be in the span of the columns of $A$.

   Please also look into Note 3 Subsection 3.3 for the definition of span.

2. Inverse of a Matrix-Matrix Product

**Learning Goal:** This problem aims to familiarize you with the properties of inverse and related proof techniques.

Prove that if a matrix-matrix product $AB$ is invertible, the inverse will be equal to $B^{-1}A^{-1}$. Please see Note 6: subsection 6.1.1 for properties of inverse.

**HINT:** We start again with what we know. Since $AB$ is invertible, we know that an inverse exists, i.e.

$$(AB)(AB)^{-1} = I$$

$$(AB)^{-1}(AB) = I$$

3. Functional Pumps

**Learning Goal:** The goal of this problem is to present a state transition diagram and guide students to understand the meaning of a state transition matrix and its applications. Please review Note 5: Section 5.1 to understand this problem better.

Take a look at this functional pump:
(a) What do the rows in a functional pump represent? What do the columns represent?

(b) Analyze the pump above and write the first column of the state transition matrix. Use the state vector:
\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]

Repeat this process for each of the reservoirs in this diagram.

(c) Is this system conserved? Why or why not? Please review Note 5: Section 5.1.4 to understand this problem better.

(d) Given that the initial reservoir volume, \( v[0] \), is
\[
\begin{bmatrix}
  1 \\
  1 \\
  1
\end{bmatrix}
\]
determine the amount of water in each of the reservoirs after turning the system on \( n \) number of times. Please review Note 5: Section 5.1.7 to understand this problem better.

   i. Turn the system on once.
   ii. Turn the system on twice.
   iii. What is another way to find \( v[2] \) if you could only multiply one state transition matrix into the initial state once?

(e) (PRACTICE) Let us model a system with reservoir states \( x_1, x_2, a, b, y_1, y_2, y_3 \) as given by the diagram below:
Write the state transition matrix for the above state transition diagram. Use the state vector:

$$\begin{bmatrix}
x_1 \\
x_2 \\
a \\
b \\
y_1 \\
y_2 \\
y_3
\end{bmatrix}$$

4. Invertibility and Row Operations

**Learning Goal:** This question introduces, through the context of finding a given matrix’s inverse, how we can represent different types of transformations and row operations with matrices. Also, we will see whether the order of applying matrix operations matters. Please review Section 2.1 of Note 2B and Section 6.1 of Note 6 to understand the problem better.

(a) Say we have a matrix \(M \in \mathbb{R}^{3 \times n}\) and a matrix \(A\), which are given by:

\[
M = \begin{bmatrix}
m_1^T \\
m_2^T \\
m_3^T
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1}{5} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

If we left multiply \(M\) by \(A\) (computing the product \(AM\)), what kind of row operation is done on \(M\)?

(b) We have the matrix \(M \in \mathbb{R}^{3 \times n}\) as before, as well as the matrix \(B\), which is given by:

\[
B = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

If we left-multiply \(M\) by \(B\), what kind of row operation is done on \(M\)?

(c) We have the matrix \(M \in \mathbb{R}^{3 \times n}\) as before, as well as the matrix \(C\), given by:

\[
C = \begin{bmatrix}
1 & 0 & 0 \\
3 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

What kind of row operation is done on \(M\)?

(d) What happens when we apply the transformations (row operations) described in parts (a), (b), and (c) to the matrix \(Q = \begin{bmatrix}
0 & 0 & 1 \\
-15 & 5 & 0 \\
1 & 0 & 0
\end{bmatrix}\)?

(e) Multiply the matrices for each of the transformations in parts (a), (b), and (c), so that the are applied in this order: (a) is applied first and (c) is applied last. Call the resulting matrix \(D\). What happens when you left multiply the \(Q\) from part (d) by \(D\)? What about right multiplying \(Q\) by \(D\)? What kind of matrix is \(D\) in relation to \(Q\)?
(f) Are there a set of transformations we can apply to \( Q = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix} \) to make it the identity? If so, what are they? If not, why is it not possible?