CSM 16A Designing Information Devices and Systems I
Fall 2020 Week 4

1. Eigen Calculation

Learning Goal: The goal of this problem is to practice mechanically calculating eigenvalues and finding their corresponding eigenvectors/eigenspaces.

Relevant Notes: **Note 9 Sections 9.4 and 9.6** cover the process of finding eigenvalue-eigenvector pairs.

(a) Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Also find the eigenspaces.

(b) Find the eigenvectors for matrix $\mathbf{A}$ given that we know that $\lambda_1 = 4, \lambda_2 = \lambda_3 = -2$ and that

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Also find the eigenspaces.

(c) Find the eigenvalues for matrix $\mathbf{A}$ given that we know that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ are the eigenvectors of $\mathbf{A}$, and that

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

2. Give Me That Eigenvalue

Learning Goal: The goal of this problem is to build intuition behind identifying eigenvalues and eigenvectors.

Relevant Notes: Note 9 Section 9.2 reinforces the concepts of eigenvalues and eigenvectors.

For each of the parts of this problem, identify what are the eigenvalues of the matrix or determine that they don't exist, **without doing any mechanical calculations**. Also find the corresponding eigenvector(s).

(a) What is the one eigenvalue and corresponding eigenvector of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}?$$

(b) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(c) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

(d) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{D} = \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix}?$$

(e) Consider a matrix that rotates a vector in $\mathbb{R}^2$ by $45°$ counterclockwise about the origin in a coordinate plane. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. This matrix is given as

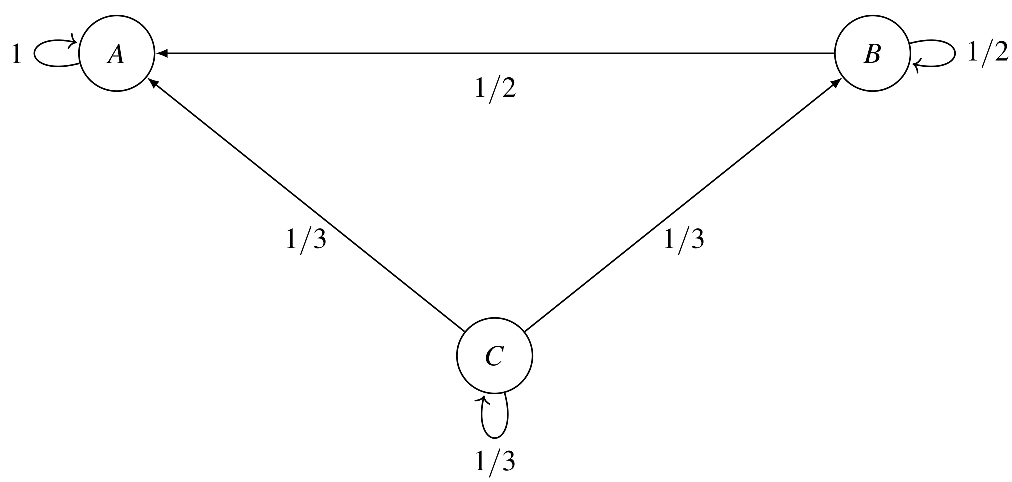
$$\mathbf{E} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

What are the eigenvalues and eigenvectors of this matrix?

3. Page Rank

Learning Goal: This problem is designed to provide insight into state transition. We will observe how the steady state depends of the eigenvalue and eigenvectors of a state-transition matrix.

Now suppose we have a network consisting of 3 websites connected as shown below. Each of the weights on the edges represent the probability of a user taking that edge.



(a) Call the transition matrix for this system **P** Write down **P** from this graph. (*Hint: Try to recall the properties of transition matrices and observe the sum of each column*).

(b) We want to rank these webpages in order of importance. Can you predict at least one of the eigenvalues of $\mathbf{P}$? Verify your predicted eigenvalue by calculation and then find the corresponding eigenvectors of $\mathbf{P}$.

(c) Now you are told that the other two eigenvalues of $\mathbf{P}$ are $\lambda_2 = \frac{1}{2}$ and $\lambda_3 = \frac{1}{3}$, and the corresponding eigenvectors are $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, respectively.
Suppose we start with just 30 users in A and no users in B and C. Can you express the initial state, $\vec{x}[0]$, as a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$?

(d) Now use the results from the previous part to express the state at time step n as a function of the eigenvectors and eigenvalues. What is the steady-state? Is the steady-state different from the initial state? Why?
Relevant Notes: **Note 9: Subsection 9.8.2** are helpful for this problem.

(e) Now suppose we start with 30 users in A, 30 users in B and no users in C. Express the initial state, $\vec{x}[0]$, as a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ and find the steady state. Is the steady-state different from the initial state? Why?

(f) Suppose that we start with 90 users evenly distributed among the websites. Without doing any calculations, can you estimate the steady-state number of people who will end up at each website?