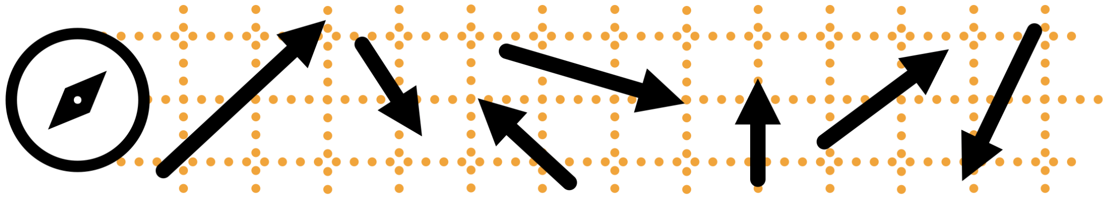
EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 1B

1. Vectors



A vector is an ordered list of numbers. For instance, a point on a plane (x,y) is a vector! We label vectors using an arrow overhead $\vec{v}$, and since vectors can live in ANY dimension of space we'll need to leave our notation general $(x,y) \rightarrow \vec{v} = (v_1, v_2, \ldots)$. Below are few more examples (the left-most form is the general definition):

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \vec{a} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3 \qquad \vec{b} = \begin{bmatrix} 2.4 \\ 5.3 \end{bmatrix} \in \mathbb{R}^2$$

Just to unpack this a bit more, $\vec{b} \in \mathbb{R}^3$ in english means "vector $\vec{b}$ lives in 3-Dimensional space".

- The $\in$ symbol literally means "in"
- The $\mathbb{R}$ stands for "real numbers" (FUN FACT: $\mathbb{Z}$ means "integers" like $-2, 4, 0, \ldots$)
- The exponent $\mathbb{R}^{n \leftarrow}$ indicates the dimension of space, or the amount of numbers in the vector.

One last thing: it is standard to write vectors in column-form, like seen with $\vec{a}, \vec{b}, \vec{x}$ above. We call these *column vectors*, in contrast to horizontally written vectors which we call *row vectors*.

Okay, let's dig into a few examples:

(a) Which of the following vectors live in $\mathbb{R}^2$ space?

$$i. \begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad ii. \begin{bmatrix} 5 \\ 0 \\ 3 \\ 5 \end{bmatrix} \qquad iii. \begin{bmatrix} -4.76 \\ 1.32 \\ 0.01 \end{bmatrix} \qquad iv. \begin{bmatrix} -20 \\ 100 \end{bmatrix}$$

Solution/Answer:
i. Yes ii. No iii. No iv. Yes
Remember $\mathbb{R}^2$ means 2D space, which hosts vectors with 2 terms.
We count and see only i. and iv. have 2 terms.

(b) Graphically show the vectors (either in a sketch with axes, or a plot on a computer):

$$i. \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad ii. \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Solution/Answer: Although these vectors look similar, remember that the ordering matters!

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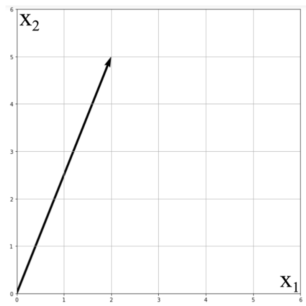
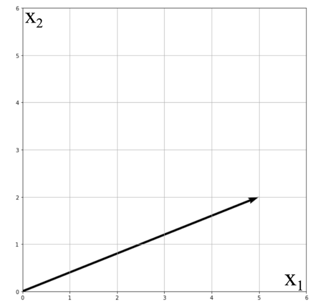
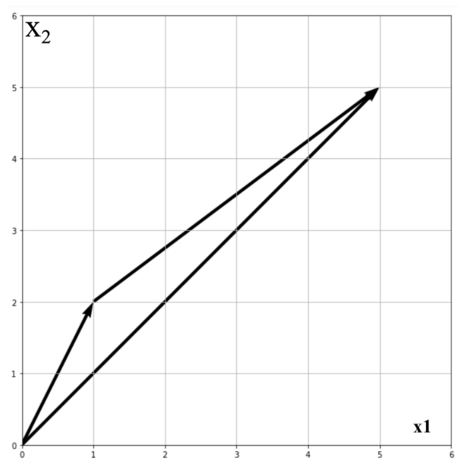




(c) Compute the sum $\vec{a}+\vec{b}=\vec{c}$ from the vectors below, and then graphically sketch or plot these vectors. (show them in a way that forms a triangle; also is there only one possible triangle?)

$$\vec{a} = \begin{bmatrix}1\\2\end{bmatrix} \qquad \vec{b} = \begin{bmatrix}4\\3\end{bmatrix}$$

Solution/Answer:



Computation is done element-wise:

$$\vec{c} = \vec{a}+\vec{b} = \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}$$

2. **Solving Systems of Equations** A system of linear equations can either have one solution, an infinite number of solutions, or no solution at all. For the following systems of equations, state whether there is a unique solution, no solution, or an infinite number of solutions. If there are an infinite number of solutions give one possible solution.

(a) Solve the following system. How many solutions does it have?

$$\begin{aligned} x + y &= 4 \\ x - y &= 2 \end{aligned}$$

Solution/Answer: Adding the two equations together, we get

$$2x = 4+2 = 6$$

This gives $x = 3$. Substituting back into the first equation, we get

$$3 + y = 4,$$

which gives us $y = 1$.
So there is a unique solution, $x = 3$ and $y = 1$.

(b) Now write the system in augmented matrix form:

$$x + y = 4 \quad (1)$$
$$x - y = 2 \quad (2)$$

Solution/Answer:
Augmented matrix form is just a shorthand notation for representing the system. You can write:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right]$$

(c) Once in augmented matrix form we can use a systematic procedure called Gaussian Elimination to solve the system of equations. See what solution you get using Gaussian elimination.

Solution/Answer:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right]$$

We can subtract the first equation from the second equation to eliminate the x variable.

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1-1 & -1-1 & 2-4 \end{array}\right].$$

The first row here represents the equation (1).
$x + y = 4$.
The second row represents the equation (2) - (1):
$(x - y) - (x + y) = 2 - 4$.
The augmented matrix form represents this by grouping the x terms and the y terms together, i.e.
$(x - x) - (y + y) = 2 - 4$,
or
$0 - 2y = -2$.
The augmented matrix is written as:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -2 & -2 \end{array}\right],$$

From here we read off that $y = 1$.
Now, we can back substitute into (1) to get that $x = 3$.

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(d) Now consider the system

$$7x + y = 7 \quad (3)$$
$$42x + 6y = 42. \quad (4)$$

How many solutions does it have? Solve it first using any method, then write it as an augmented matrix and try to solve it.

Solution/Answer:
We can see that the first equation gives the second equation when multiplied by 6. But how would we do this systematically? We first write the matrix:

$$\left[\begin{array}{cc|c} 7 & 1 & 7 \\ 42 & 6 & 42 \end{array}\right]$$

Then, taking six times the first row and subtracting it from the second gives:

$$\left[\begin{array}{cc|c} 7 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right].$$

So the second equation reduces to $0 = 0$. Therefore there are infinitely many solutions. How could write out all the infinitely many solutions.
For every value of y we choose we can write a value of x.
So let $y = a$, where a is some arbitrary real number. We can use a shorthand notation to write this as $a \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and $\in$ means in.
Substituting this into the first equation we get:
$7x + a = 7$,
so
$x = (7 - a)/7$. This is sometimes called a parametric form of the solution with parameter a.
If you wanted to write the set of solutions as a vector, you could write all vectors of the form

$$\begin{bmatrix} (7-a)/7 \\ a \end{bmatrix}$$

are solutions.

(e) Now consider the system

$$7x + y = 7 \quad (5)$$
$$42x + 6y = 42 \quad (6)$$
$$7x + y = 6 \quad (7)$$

How many solutions does it have? Solve it first using any method, then write it as an augmented matrix and try to solve it.

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Solution/Answer:

Here we see that (6) is a multiple of (5), however, (5) and (7) are inconsistent. So no solution exists. But how would we do this systematically? We first write the matrix:

$$\left[\begin{array}{cc|c} 7 & 1 & 7 \\ 42 & 6 & 42 \\ 7 & 1 & 6 \end{array}\right]$$

Then, taking six times the first row and subtracting it from the second gives:

$$\left[\begin{array}{cc|c} 7 & 1 & 7 \\ 0 & 0 & 0 \\ 7 & 1 & 6 \end{array}\right].$$

The subtracting the first row from the second gives:

$$\left[\begin{array}{cc|c} 7 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array}\right].$$

Here the last row represents the equation $0y = -1$, which we know cannot have a solution. Hence, in this case the system has no solutions.

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