EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 3B

1. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

For each matrix multiplication problem, *if the product exists*, find the product by hand. Otherwise, explain why the product does not exist.

(a) **A B** **Answer:** A is a 1×2 vector and B_1 is a 2×1 vector, so the product exists!
$\mathbf{AB} = 1 \cdot 3 + 4 \cdot 2 = 11$

(b) **C D** **Answer:** Since both **C** and **D** are 2×2 matrices, the product exists and is a 2×2 matrix.
$$\mathbf{CD} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 4 \cdot 2 & 1 \cdot 2 + 4 \cdot 1 \\ 2 \cdot 3 + 3 \cdot 2 & 2 \cdot 2 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}.$$

(c) **D C** **Answer:** Since both **C** and **D** are 2×2 matrices, the product exists and is a 2×2 matrix.
$$\mathbf{DC} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 2 \cdot 2 & 3 \cdot 4 + 2 \cdot 3 \\ 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 4 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}.$$

(d) **C E** **Answer:** Since **C** is a 2×2 matrix and **E** is a 2×4 matrix, the product exists and is a 2×4 matrix.
$$\mathbf{CE} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 4 \cdot 4 & 1 \cdot 9 + 4 \cdot 3 & 1 \cdot 5 + 4 \cdot 2 & 1 \cdot 7 + 4 \cdot 2 \\ 2 \cdot 1 + 3 \cdot 4 & 2 \cdot 9 + 3 \cdot 3 & 2 \cdot 5 + 3 \cdot 2 & 2 \cdot 7 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}.$$

(e) **F E** **Answer:**
Since **E** is a 2×4 matrix and **F** is a 4×3 matrix, the product does not exist.
This is because the number of columns in the first matrix (**F**) should match the number of rows in the second matrix (**E**) for this product to be defined.

(f) **E F** **Answer:**
Since **E** is a 2×4 matrix and **F** is a 4×3 matrix, the product exists and is a 2×3 matrix.

$$\mathbf{EF} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1\cdot 5+9\cdot 6+5\cdot 4+7\cdot 3 & 1\cdot 5+9\cdot 1+5\cdot 1+7\cdot 2 & 1\cdot 8+9\cdot 2+5\cdot 7+7\cdot 2 \\ 4\cdot 5+3\cdot 6+2\cdot 4+2\cdot 3 & 4\cdot 5+3\cdot 1+2\cdot 1+2\cdot 2 & 4\cdot 8+3\cdot 2+2\cdot 7+2\cdot 2 \end{bmatrix}$$
$$= \begin{bmatrix} 100 & 33 & 75 \\ 52 & 29 & 56 \end{bmatrix}$$

(g) **G H** (Practice on your own) **Answer:** Since **G** and **H** are both 3×3 matrices, the product exists and is another 3×3 matrix.

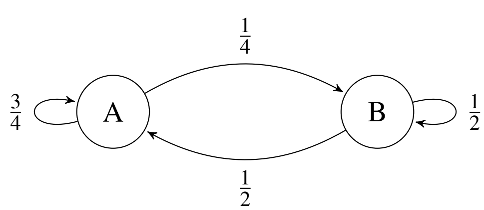
$$\mathbf{GH} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 8\cdot 5+1\cdot 1+6\cdot 2 & 8\cdot 3+1\cdot 8+6\cdot 3 & 8\cdot 4+1\cdot 2+6\cdot 5 \\ 3\cdot 5+5\cdot 1+7\cdot 2 & 3\cdot 3+5\cdot 8+7\cdot 3 & 3\cdot 4+5\cdot 2+7\cdot 5 \\ 4\cdot 5+9\cdot 1+2\cdot 2 & 4\cdot 3+9\cdot 8+2\cdot 3 & 4\cdot 4+9\cdot 2+2\cdot 5 \end{bmatrix} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}.$$

(h) **H G** (Practice on your own) **Answer:** Since **H** and **G** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{GH} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 5\cdot 8+3\cdot 3+4\cdot 4 & 5\cdot 1+3\cdot 5+4\cdot 9 & 5\cdot 6+3\cdot 7+4\cdot 2 \\ 1\cdot 8+8\cdot 3+2\cdot 4 & 1\cdot 1+8\cdot 5+2\cdot 9 & 1\cdot 6+8\cdot 7+2\cdot 2 \\ 2\cdot 8+3\cdot 3+5\cdot 4 & 2\cdot 1+3\cdot 5+5\cdot 9 & 2\cdot 6+3\cdot 7+5\cdot 2 \end{bmatrix} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}.$$

2. Transition Matrix

Suppose we have a network of pumps as shown in the diagram below. Let us describe the state of A and B using a state vector $\vec{x}[n] = \begin{bmatrix} x_A[n] \\ x_B[n] \end{bmatrix}$ where $x_A[n]$ and $x_B[n]$ are the states at time-step n.



(a) Find the state transition matrix S, such that $\vec{x}[n+1] = S\,\vec{x}[n]$.

Answer: We can write the following equations by examining the state transition diagram:

$$\begin{aligned} x_A[n+1] &= (3/4)\, x_A[n] + (1/2)\, x_B[n] \\ x_B[n+1] &= (1/4)\, x_A[n] + (1/2)\, x_B[n] \end{aligned}$$

From here, we can directly write down the state transition matrix as:

$$S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

Notice that the columns of S sum to 1, which ensures we have a physical system that satisfies conservation.

(b) Let us now find the matrix S^{-1} such that we can recover the previous state $\vec{x}[n-1]$ from $\vec{x}[n]$. Specifically, solve for S^{-1} such that $\vec{x}[n-1] \;=\; S^{-1}\,\vec{x}[n]$.

Answer: We can use Gaussian elimination to solve for the matrix S^{-1}, i.e. inverse of the matrix S that we just found:

$$\left[\begin{array}{cc|cc} 3/4 & 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{4}{3}R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow -4R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right] \xrightarrow{R_2 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \xrightarrow{R_2 \leftarrow -\frac{3}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 3 \end{array}\right].$$

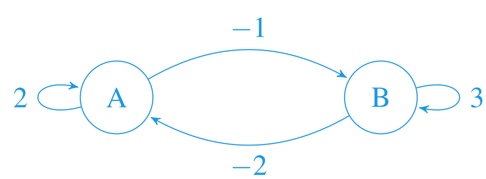
Therefore:

$$S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

Note that the columns of S^{-1} still sum to 1 despite matrix elements exceeding 1 and going negative

(c) Now draw the state transition diagram that corresponds to the S^{-1} that you just found.

Answer:



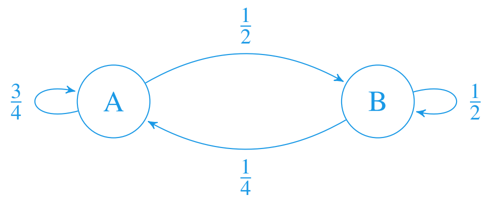
We can write the following equations by examining the state transition diagram:

$$\begin{aligned} x_A[n-1] &= 2\,x_A[n] \;-\; 2x_B[n] \\ x_B[n-1] &= -x_A[n] \;+\; 3\,x_B[n] \end{aligned}$$

Because the matrix S^{-1} is an inverse matrix, it can be thought of as the matrix that *turns back time* for the pump system. **Although it is non-physical**, the weights that have an absolute value greater than 1 can be thought of as "generating" water, and the weights that have negative weight can be thought of as "destroying" water. However, note that the outflow weights of each node still sum to 1 (i.e. the columns of S^{-1} still sum to 1). This means that in total all of the water is being conserved during the transition between time steps, even when time is reversed.

(d) Redraw the diagram from the first part of the problem, but now with the directions of the arrows reversed. Let us call the state transmission matrix of this "reversed" state transition diagram T. Does $T = S^{-1}$?

Answer:



After drawing the "reversed" state transition diagram, we can write the following equations:

$$x_A[n+1] = (3/4)\, x_A[n] + (1/4)\, x_B[n]$$
$$x_B[n+1] = (1/2)\, x_A[n] + (1/2)\, x_B[n]$$

From here, we can directly write down the state transition matrix as:

$$T = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Note that $T \neq S^{-1}$. What we have actually found is that T is equal to the *transpose* of S, denoted by $S^{\top}$ (the superscript $\top$ denotes the transpose of a matrix). The transpose of a matrix is when its rows become its columns. In general, a matrix's inverse and its transpose are not equal to each other.

(e) Suppose we start in the state $\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$. Compute the state vector after 2 time-steps $\vec{x}[3]$.

Answer: There are two way to approach this problem:

i. Compute states successively $\vec{x}[2] = S\,\vec{x}[1]$, then $\vec{x}[3] = S\,\vec{x}[2]$
ii. Compute directly by $\vec{x}[3] = S\,S\,\vec{x}[1]$

They are both equivalent thanks to the fact that matrix multiplication is associative:

$$\vec{x}[3] = S(S\,\vec{x}[1]) = (S\,S)\,\vec{x}[1]$$

We start with method i.

$$\vec{x}[2] = S\,\vec{x}[1] = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \cdot 12/4 + 12/2 \\ 1 \cdot 12/4 + 12/2 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$$

$$\vec{x}[3] = S\,\vec{x}[2] = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \cdot 15/4 + 9/2 \\ 1 \cdot 15/4 + 9/2 \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix} \qquad \square$$

Alternatively we can use method ii.

$$\vec{x}[3] = S\,S\,\vec{x}[1] \quad \rightarrow \quad S^2 = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 9/16 + 1/8 & 3/8 + 1/4 \\ 3/16 + 1/8 & 1/8 + 1/4 \end{bmatrix} = \begin{bmatrix} 11/16 & 5/8 \\ 5/16 & 3/8 \end{bmatrix}$$

$$\vec{x}[3] = S^2\,\vec{x}[1] = \begin{bmatrix} 11/16 & 5/8 \\ 5/16 & 3/8 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 11 \cdot 3/4 + 5 \cdot 6/4 \\ 5 \cdot 3/4 + 3 \cdot 6/4 \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix} \qquad \square$$

(f) **(Challenge practice problem)** Given our starting state from the previous problem, what happens if we look at the state of the network after a lot of time steps? Specifically which state are we approaching, as defined below?

$$\vec{x}_{final} = \lim_{n\to\infty} \vec{x}[n]$$

Note that the final state needs to be what we call a *steady state*, meaning $S\,\vec{x}_{final} = \vec{x}_{final}$.

Also what can you say about $x_A[n] + x_B[n]$?

Use information from both of these properties to write out a new system of equations and solve for $\vec{x}_{final}$.

Answer:

We first use the fixed point property to yield an equation relating x_A and x_B for $\vec{x}_{final}$. To see this we subtract $\vec{x}_{final}$ from the fixed point equation to get $S\,\vec{x}_{final} - \vec{x}_{final} = 0$.

To make this easier we recognize that $\vec{x}_{final}$ can be written as the result of the *identity* matrix operation $I\,\vec{x}_{final} = \vec{x}_{final}$.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \rightarrow \quad I\,\vec{x}_{final} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \vec{x}_{final}$$

Thus our expression becomes a system of equations: $(S - I)\vec{x}_{final} = \vec{0}$. Let's see if we can solve it!

$$(S-I)\vec{x}_{final} = \vec{0} \quad \rightarrow \quad \left[\begin{array}{cc|c} 3/4-1 & 1/2-0 & 0 \\ 1/4-0 & 1/2-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 1/4 & -1/2 & 0 \end{array}\right]$$

With two row operations ($R_2 \rightarrow R_2 + R_1$ and then $R_1 \rightarrow -4R_1$) we see there are infinite solutions, but at least attain one condition x_A and x_B must satisfy for $\vec{x}_{final}$:

$$\left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 1/4 & -1/2 & 0 \end{array}\right] \overset{R_2\rightarrow R_2+R_1}{\Longrightarrow} \left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \overset{R_1\rightarrow -4R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_A - 2x_B = 0$$

Second, we can look at our computed states from before and notice a pattern:

$$\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix} \rightarrow \vec{x}[2] = \begin{bmatrix} 15 \\ 9 \end{bmatrix} \rightarrow \vec{x}[3] = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix}$$

Most notably $x_A[1] + x_B[1] = 12 + 12 = 24$, $x_A[2] + x_B[2] = 15 + 9 = 24$, and $x_A[3] + x_B[3] = (63 + 33)/4 = 24$. This is no coincidence; it is the result of our conservation property of the network seen by the columns of S summing to 1. So generally $x_A[n] + x_B[n] = 24$ for any n, including the final state.

Now we have two expressions and can solve for $\vec{x}_{final}$:

$$\begin{array}{c} x_A - 2x_B = 0 \\ x_A + x_B = 24 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 1 & 24 \end{array}\right] \overset{R_2 \to R_2 - R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 3 & 24 \end{array}\right] \rightarrow \begin{array}{c} x_B = 8 \\ x_A = 0 + 2(8) \rightarrow x_A = 16 \end{array}$$

Thus the final state of the system is $\vec{x}_{final} = \begin{bmatrix} 16 \\ 8 \end{bmatrix}$. □