EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 4A

1. Mechanical Inverses

In each part, determine whether the inverse of $\mathbf{A}$ exists. If it exists, find it.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer:

We use Gaussian elimination (also known as the Gauss-Jordan method):

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \underbrace{\Rightarrow}_{R_2 \leftarrow \frac{1}{9}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$.

(b) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Answer: We can again use the Gauss-Jordan method:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \underbrace{\Rightarrow}_{R_1 \leftarrow \frac{1}{a}R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\underbrace{\Rightarrow}_{R_2 \leftarrow R_2 - cR_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{c}{a}b & -\frac{c}{a} & 1 \end{array}\right]$$

$$\underbrace{\Rightarrow}_{R_2 \leftarrow \frac{1}{d-\frac{c}{a}b}R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-\frac{c}{a}}{d-\frac{c}{a}b} & \frac{1}{d-\frac{c}{a}b} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\underbrace{\Rightarrow}_{R_1 \leftarrow R_1 - \frac{b}{a}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{b}{a}\frac{c}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

Therefore, we get that

$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

This is a known formula which, if you find useful, you can use for 2x2 matrices.

(c) (PRACTICE)

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$$

Answer:

We use Gaussian elimination:

$$\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \overset{R_1 \leftarrow R_2}{\Rightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 5 & 4 & 1 & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow -5R_1 + R_2}{\Rightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -5 \end{array}\right] \overset{R_1 \leftarrow R_1 + R_2}{\Rightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & -1 & 1 & -5 \end{array}\right]$$

$$\overset{R_2 \leftarrow -R_2}{\Rightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 5 \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$.

(d) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

Answer:

We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftarrow \frac{1}{5}R_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_2 \leftarrow \frac{1}{2}R_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow R_2 - R_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \overset{R_3 \leftarrow R_3 + R_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array}\right].$$

While row-reducing, we notice that the second column doesn't have a pivot (and that there is also a row of zeros). Therefore, no inverse exists.

(e) (PRACTICE)

$$\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix}$$

Answer:

We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overbrace{\Rightarrow}^{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overbrace{\Rightarrow}^{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right]$$

$$\overbrace{\Rightarrow}^{R_2 \leftarrow -R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right] \overbrace{\Rightarrow}^{R_3 \leftarrow -R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

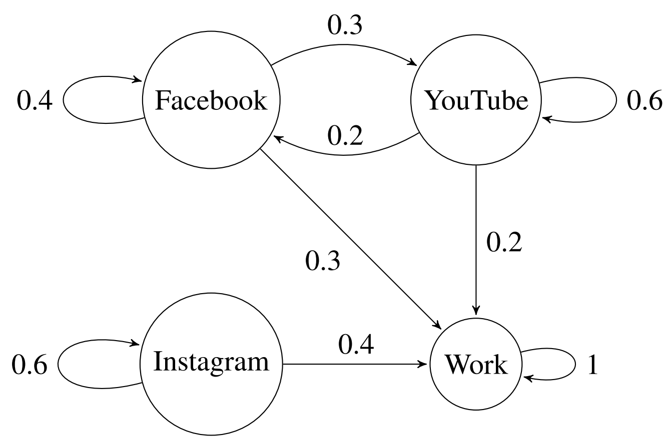
$$\overbrace{\Rightarrow}^{R_2 \leftarrow R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{2}{5} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\overbrace{\Rightarrow}^{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{5} & 2 & 1 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{4}{5} & 2 & 1 \\ \frac{2}{5} & -\frac{1}{2} & -1 \\ \frac{1}{5} & -\frac{1}{2} & 0 \end{bmatrix}$.

2. Social Media

As a tech-savvy Berkeley student, the distractions of social media are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Facebook or YouTube? How do other students manage to get stuff done and balance pursuing Insta-fame? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if 100 students are on Facebook, in the next timestep, 30 of them will click on a link and move to YouTube.



(a) Derive the corresponding transition matrix.

Answer:

Let us define $x_F[n]$ as the number of students on Facebook at timestep n, $x_Y[n]$ as the number of students on YouTube at timestep n, $x_I[n]$ as the number of students on Instagram at timestep n, and $x_W[n]$ as the number of students working at timestep n. Let us now explicitly write the equations that we can then use to determine the state transition matrix.

$x_F[n+1] = 0.4x_F[n] + 0.2x_Y[n]$
$x_Y[n+1] = 0.3x_F[n] + 0.6x_Y[n]$
$x_I[n+1] = 0.6x_I[n]$
$x_W[n+1] = 0.3x_F[n] + 0.2x_Y[n] + 0.4x_I[n] + x_W[n]$

Let $\vec{x}[n] = \begin{bmatrix} x_F[n] \\ x_Y[n] \\ x_I[n] \\ x_W[n] \end{bmatrix}$.

We can now solve for the state transition matrix A such that:
$\vec{x}[n+1] = A\vec{x}[n]$.
A is therefore equal to:

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}$$

(b) There are 1500 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 700 EECS16A students on Facebook, 450 on YouTube, 200 on Instagram, and 150 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

Answer:

$$\begin{bmatrix} 370 \\ 480 \\ 120 \\ 530 \end{bmatrix}$$

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

Answer:

Since each column's sum is equal to 1, the system is conservative. This means that we aren't losing students after each time step.

(d) You want to predict how many students will be on each website n timesteps in the future. How would you formulate that mathematically? Without working it out, can you predict roughly how many students will be in each state 100 timesteps in the future?

Answer:

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}^n \vec{x}[0] = \vec{x}[n]$$

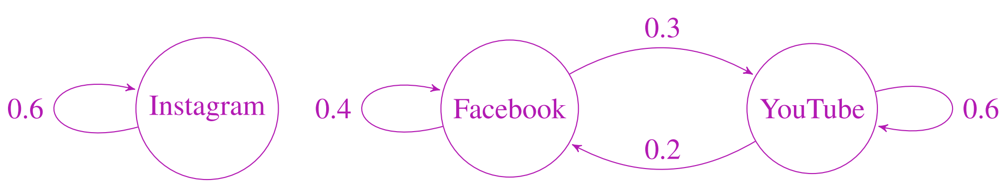
All of them will be working! Yay! With this particular system, 'Work' is called a 'final accepting state' or an 'absorbing state.' This means all the students, after jumping around and being distracted for some amount of time, will eventually end up working. Why is this? 'Work' is the only state where 100% of students who are working remain working. So as time passes, a student has some probability of landing in Work but 0 probability of leaving Work. If you actually calculate $\mathbf{A}^{100}$, you'll see that all the "mass" in the problem transfers to the bottom row, numerically reflecting the fact that 'Work' is absorbing all of the students.

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}^{100} = \begin{bmatrix} 6.83599885 \cdot 10^{-13} & 8.30745059 \cdot 10^{-13} & 0 & 0 \\ 1.24611759 \cdot 10^{-12} & 1.51434494 \cdot 10^{-12} & 0 & 0 \\ 0 & 0 & 6.53318624 \cdot 10^{-23} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The above was calculated using IPython notebook.

(e) **Challenging Practice Problem:** Suppose, instead of having 'Work' as an explicit state, we assume that any student not on Facebook/Youtube/Instagram is working. Work is like the "void," and if a student is "leaked" from any of the other states, we assume s/he has gone to work and will never come back. How would you reformulate this problem? Redraw the figure and rewrite the appropriate transition matrix. What are the major differences between this problem and the previous one?

Answer:



$$\begin{bmatrix} 0.4 & 0.2 & 0 \\ 0.3 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}$$

Since we don't track students who have gone to work, the entries in the columns of the state transition matrix no longer sum to 1. Because they sum to a number less than 1, the system is not conservative and eventually all students will disappear from the system.

$$\begin{bmatrix} 0.4 & 0.2 & 0 \\ 0.3 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}^{100} = \begin{bmatrix} 0.0684 \cdot 10^{-11} & 0.0831 \cdot 10^{-11} & 0 \\ 0.1246 \cdot 10^{-11} & 0.1514 \cdot 10^{-11} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Practice: Column Spaces and Null Spaces Intro

- The **column space** is the possible outputs of a transformation/function/linear operation. It is also the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?
ii. What is the null space of $\mathbf{A}$? What is its dimension?
iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?
iv. Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.

(b) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: $\mathbb{R}^2$

Null space: span $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

Yes, the two column spaces are the same as the column span $\mathbb{R}^2$.
This is a basis for $\mathbb{R}^2$.

(c) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.