EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 4B

1. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?
ii. What is the null space of $\mathbf{A}$? What is its dimension?
iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?
iv. Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

Null space: $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.
Not a basis for $\mathbb{R}^2$.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Null space: $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: $\mathbb{R}^2$

Null space: $\text{span}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

The two column spaces are the same as the column span $\mathbb{R}^2$.
This is a basis for $\mathbb{R}^2$.

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

Answer:

i. The columnspace of the columns is $\mathbb{R}^2$. The columns of $\mathbf{A}$ do not form a basis for $\mathbb{R}^2$. This is because the columns of $\mathbf{A}$ are linearly dependent.

ii. The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation $\mathbf{A}\vec{x} = \vec{0}$ by performing Gaussian elimination on $\mathbf{A}$. We start by performing Gaussian elimination on matrix $\mathbf{A}$ to get the matrix into upper-triangular form.

$$\begin{aligned} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \text{reduced row echelon form} \end{aligned}$$

$$\begin{aligned} &x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0 \\ &x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0 \\ &x_3 \text{ is free and } x_4 \text{ is free} \end{aligned}$$

Now let $x_3 = s$ and $x_4 = t$. Then we have:

$$\begin{aligned} x_1 + \frac{1}{2}s - \frac{7}{2}t = 0 \\ x_2 + \frac{5}{2}s + \frac{1}{2}t = 0 \end{aligned}$$

Now writing all the unknowns (x_1, x_2, x_3, x_4) in terms of the dummy variables:

$$\begin{aligned} &x_1 = -\frac{1}{2}s + \frac{7}{2}t \\ &x_2 = -\frac{5}{2}s - \frac{1}{2}t \\ &y = s \\ &z = t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{7}{2}t \\ -\frac{5}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{5}{2}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2}t \\ -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

So every vector in the nullspace of $\mathbf{A}$ can be written as follows:

$$\text{Nullspace}(\mathbf{A}) = s\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the nullspace of $\mathbf{A}$ is

$$\text{span}\left\{\begin{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\end{bmatrix}\right\}$$

$\mathbf{A}$ has a 2-dimensional null space.

iii. In this case, the column space of the row reduced matrix is also $\mathbb{R}^2$, but this need not be true in general.

iv. No, the columns of $\mathbf{A}$ do not form a basis for $\mathbb{R}^2$.

2. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? If the vectors do not form a basis for $\mathbb{R}^3$, can they be thought of as a basis for some other vector space? If so, write an expression describing this vector space.

$$V_1 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$$

Answer:

- V_1: The vectors are linearly independent, but they are not a basis for $\mathbb{R}^3$, because you cannot construct all vectors in $\mathbb{R}^3$ using these vectors. Instead, they are a basis for some 2-dimensional subspace of $\mathbb{R}^3$. This subspace can be described by $\text{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$.
- V_2: Yes, the vectors are linearly independent and will form a basis for $\mathbb{R}^3$. To check that the vectors are linearly independent, you should do Gaussian Elimination of the matrix of the columns: $\begin{bmatrix}1 & 1 & 1\\1 & 0 & 0\\1 & 1 & 0\end{bmatrix}$. Check that you can get all the way to identity, i.e. the system has a unique solution.
- V_3: No, $\vec{v}_2 + \vec{v}_3 = \vec{v}_1$, so the vectors are linearly dependent. Hence, they cannot form a basis for any vector space of any dimension.

3. (Optional Practice) Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ where } c,d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

Answer:

Yes, V is a subspace of $\mathbb{R}^3$. We will *prove this* by using the definition of a subspace.

First of all, note that V is a subset of $\mathbb{R}^3$ – all elements in V are of the form $\begin{bmatrix}c+d\\c\\c+d\end{bmatrix}$, which is a 3-dimensional real vector.

Now, consider two elements $\vec{v}_1, \vec{v}_2 \in V$ and $\alpha \in \mathbb{R}$.

This means that there exists $c_1, d_1 \in \mathbb{R}$, such that $\vec{v}_1 = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + d_1\begin{bmatrix}1\\0\\1\end{bmatrix}$. Similarly, there exists $c_2, d_2 \in \mathbb{R}$, such that $\vec{v}_2 = c_2\begin{bmatrix}1\\1\\1\end{bmatrix} + d_2\begin{bmatrix}1\\0\\1\end{bmatrix}$.

Now, we can see that

$$\vec{v}_1 + \vec{v}_2 = (c_1+c_2)\begin{bmatrix}1\\1\\1\end{bmatrix} + (d_1+d_2)\begin{bmatrix}1\\0\\1\end{bmatrix},$$

so $\vec{v}_1 + \vec{v}_2 \in V$.

Also,

$$\alpha\vec{v}_1 = (\alpha c_1)\begin{bmatrix}1\\1\\1\end{bmatrix} + (\alpha d_1)\begin{bmatrix}1\\0\\1\end{bmatrix},$$

so $\alpha\vec{v}_1 \in V$.

Furthermore, we observe that the zero vector is contained in V, when we set $c = 0$ and $d = 0$.

We have thus identified V as a subset of $\mathbb{R}^3$, shown both of the no escape (closure) properties (closure under vector addition and closure under scalar multiplication), as well as the existence of a zero vector, so V is a subspace of $\mathbb{R}^3$.

It's important to note that satisfying the subset property and the two forms of closure additionally implies this subspace V also satisfies the axioms of a vector space, and therefore is definitionally also a vector space.