1. Identifying a Subspace: Proof

Is the set
\[ V = \left\{ \vec{v} \mid \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{where } c, d \in \mathbb{R} \right\} \]
a subspace of \( \mathbb{R}^3 \)? Why or why not?

**Answer:**
Yes, \( V \) is a subspace of \( \mathbb{R}^3 \). We will prove this by using the definition of a subspace.

First of all, note that \( V \) is a subset of \( \mathbb{R}^3 \) – all elements in \( V \) are of the form \( \begin{bmatrix} c + d \\ c \\ c + d \end{bmatrix} \), which is a 3-dimensional real vector.

Now, consider two elements \( \vec{v}_1, \vec{v}_2 \in V \) and \( \alpha \in \mathbb{R} \).

This means that there exists \( c_1, d_1 \in \mathbb{R} \), such that \( \vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \). Similarly, there exists \( c_2, d_2 \in \mathbb{R} \), such that \( \vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \).

Now, we can see that
\[ \vec{v}_1 + \vec{v}_2 = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (d_1 + d_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \]
so \( \vec{v}_1 + \vec{v}_2 \in V \).

Also,
\[ \alpha \vec{v}_1 = (\alpha c_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (\alpha d_1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \]
so \( \alpha \vec{v}_1 \in V \).

Furthermore, we observe that the zero vector is contained in \( V \), when we set \( c = 0 \) and \( d = 0 \).

We have thus identified \( V \) as a subset of \( \mathbb{R}^3 \), shown both of the no escape (closure) properties (closure under vector addition and closure under scalar multiplication), as well as the existence of a zero vector, so \( V \) is a subspace of \( \mathbb{R}^3 \).

It’s important to note that satisfying the subset property and the two forms of closure additionally implies that subspace \( V \) also satisfies the axioms of a vector space, and therefore is also a vector space.
2. Exploring Column Spaces and Null Spaces

- The **column space** is the span of the column vectors of the matrix.
- The **null space** is the set of input vectors that when multiplied with the matrix result in the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $A$? What is its dimension?
ii. What is the null space of $A$? What is its dimension?
iii. Are the column spaces of the row reduced matrix $A$ and the original matrix $A$ the same?
iv. Do the columns of $A$ span $\mathbb{R}^2$? Do they form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

**Answer:**
- Column space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
- Null space: span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same. The column space does not span $\mathbb{R}^2$ and thus are not a basis for $\mathbb{R}^2$.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

**Answer:**
- Column space: span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
- Null space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

The two column spaces are not the same. Not a basis for $\mathbb{R}^2$.

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

**Answer:**
- Column space: $\mathbb{R}^2$
- Null space: span $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
The two column spaces are the same as the column span $\mathbb{R}^2$, since they are two independent vectors. This is a basis for $\mathbb{R}^2$.

(d) \[
\begin{bmatrix}
-2 & 4 \\
3 & -6
\end{bmatrix}
\]

Answer:

Column space: span \{ \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix} \}

Null space: span \{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \}

The two column spaces are not the same. We can also see that one of the columns is a scaled version of the other, and therefore, they are linearly dependent vectors.
Not a basis for $\mathbb{R}^2$.

(e) \[
\begin{bmatrix}
1 & -1 & -2 & -4 \\
1 & 1 & 3 & -3
\end{bmatrix}
\]

Answer:

i. The columnspace of the columns is $\mathbb{R}^2$. The columns of $A$ do not form a basis for $\mathbb{R}^2$. This is because the columns of $A$ are linearly dependent.

ii. The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation $Ax = 0$ by performing Gaussian elimination on $A$. We start by performing Gaussian elimination on matrix $A$ to get the matrix into upper-triangular form.

\[
\begin{bmatrix}
1 & -1 & -2 & -4 \\
1 & 1 & 3 & -3
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & -2 & -4 \\
0 & 2 & 5 & 1
\end{bmatrix} \\
\sim \begin{bmatrix}
1 & -1 & -2 & -4 \\
0 & 1 & \frac{5}{2} & \frac{1}{2}
\end{bmatrix} \\
\sim \begin{bmatrix}
1 & 0 & \frac{1}{2} & -\frac{7}{2} \\
0 & 1 & \frac{5}{2} & \frac{1}{2}
\end{bmatrix} \text{ reduced row echelon form}
\]

\[
x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0 \\
x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0 \\
x_3 \text{ is free and } x_4 \text{ is free}
\]
Now let $x_3 = s$ and $x_4 = t$. Then we have:

\[
\begin{align*}
x_1 + \frac{1}{2}s - \frac{7}{2}t &= 0 \\
x_2 + \frac{5}{2}s + \frac{1}{2}t &= 0
\end{align*}
\]

Now writing all the unknowns $(x_1, x_2, x_3, x_4)$ in terms of the dummy variables:

\[
\begin{align*}
x_1 &= -\frac{1}{2}s + \frac{7}{2}t \\
x_2 &= -\frac{5}{2}s - \frac{1}{2}t \\
x_3 &= s \\
x_4 &= t
\end{align*}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{2}s + \frac{7}{2}t \\
-\frac{5}{2}s - \frac{1}{2}t \\
s \\
t
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{2}s \\
-\frac{5}{2}s \\
s \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{7}{2}t \\
-\frac{1}{2}t \\
0 \\
t
\end{bmatrix} = s \begin{bmatrix}
-\frac{1}{2} \\
-\frac{5}{2} \\
1 \\
0
\end{bmatrix} + t \begin{bmatrix}
\frac{7}{2} \\
-\frac{1}{2} \\
0 \\
1
\end{bmatrix}
\]

So every vector in the nullspace of $A$ can be written as follows:

$\text{Nullspace}(A) = s \begin{bmatrix}
-\frac{1}{2} \\
-\frac{5}{2} \\
1 \\
0
\end{bmatrix} + t \begin{bmatrix}
\frac{7}{2} \\
-\frac{1}{2} \\
0 \\
1
\end{bmatrix}$

Therefore the nullspace of $A$ is

$\text{span} \left\{ \begin{bmatrix}
-\frac{1}{2} \\
-\frac{5}{2} \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
\frac{7}{2} \\
-\frac{1}{2} \\
0 \\
1
\end{bmatrix} \right\}$

$A$ has a 2-dimensional null space.

iii. In this case, the column space of the row reduced matrix is also $\mathbb{R}^2$, but this need not be true in general.

iv. No, the columns of $A$ do not form a basis for $\mathbb{R}^2$. 