EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 5B

1. Steady and Unsteady States

(a) You're given the matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$. ($\vec{x}$ could describe either people or water.) Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$
ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$
iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Answer:

i. $\lambda = 1$:

$$\left[\begin{array}{c|c} \mathbf{M} - \mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{G.E.}{\rightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

ii. $\lambda = 2$

$$\left[\begin{array}{c|c} \mathbf{M} - 2\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} \frac{-3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{G.E.}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$$

iii. $\lambda = \frac{1}{2}$

$$\left[\begin{array}{c|c} \mathbf{M} - \frac{1}{2}\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array}\right] \overset{G.E.}{\rightarrow} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty}\mathbf{M}^n\vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

Answer:

$$\begin{aligned}\mathbf{M}^n\vec{x} &= \mathbf{M}^n(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3)\\ &= \alpha\mathbf{M}^n\vec{v}_1 + \beta\mathbf{M}^n\vec{v}_2 + \gamma\mathbf{M}^n\vec{v}_3\\ &= 1^n\alpha\vec{v}_1 + 2^n\beta\vec{v}_2 + \left(\frac{1}{2}\right)^n\gamma\vec{v}_3\end{aligned}$$

α	β	γ	Converges?	$\lim_{n\to\infty}\mathbf{M}^n\vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	-
0	$\neq 0$	$\neq 0$	No	-
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	-
$\neq 0$	$\neq 0$	$\neq 0$	No	-

2. Eigenvalues and Special Matrices – Visualization

As seen earlier, an eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer: Multiplying the identity matrix with any vector in $\mathbb{R}^n$ produces the same vector, that is, $\mathbf{I}\vec{x} = \vec{x} = 1\cdot\vec{x}$. Therefore, $\lambda = 1$. Since $\vec{x}$ can be any vector in $\mathbb{R}^n$, the corresponding eigenvectors are all vectors in $\mathbb{R}^n$.

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0\\ 0 & d_2 & 0 & \cdots & 0\\ 0 & 0 & d_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & d_n\end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer: Since the matrix is diagonal, multiplying the diagonal matrix with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries d_i of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(c) Conceptually, does a rotation matrix in $\mathbb{R}^2$ by angle θ have any eigenvalues $\lambda \in \mathbb{R}$? For which angles is this the case?

Answer: In a conceptual sense, there are three cases:

Rotation by $0°$: (more accurately, any integer multiple of $360°$), which yields a rotation matrix $\mathbf{R} = \mathbf{I}$: This will have one eigenvalue of $+1$ because it doesn't affect any vector ($\mathbf{R}\vec{x} = \vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

Rotation by $180°$: (more accurately, any angle of $180° + n \cdot 360°$ for integer n), which yields a rotation matrix $\mathbf{R} = -\mathbf{I}$: This will have one eigenvalue of -1 because it "flips" any vector ($\mathbf{R}\vec{x} = -\vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

Any other rotation: there aren't any real eigenvalues. The reason is, if there were any real eigenvalue $\lambda \in \mathbb{R}$ for a non-trivial rotation matrix, it means that we can get $\mathbf{R}\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$, which means that by rotating a vector, we scaled it. This is a contradiction (again, unless $\mathbf{R} = \mathbf{I}$). Refer to Figure 1 for a visualization.

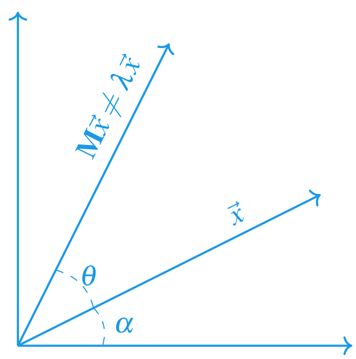

Figure 1: Rotation will never scale any non-zero vector (by a real number) unless it is rotation by an integer multiple of $360°$ (identity matrix) or the rotation angle is $\theta = 180° + n \cdot 360°$ for any integer n ($-\mathbf{I}$).

(d) Now let us mechanically compute the eigenvalues of the rotation matrix in $\mathbb{R}^2$. Does it agree with our findings above? As a refresher, the rotation matrix $\mathbf{R}$ has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Answer: Using our known determinant formula for 2x2 matrices $\det(A) = ad - bc$ we can compute the characteristic polynomial

$$\det(\mathbf{R} - \lambda\mathbf{I}) = \det \begin{bmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{bmatrix} = \cos(\theta)^2 + \sin(\theta)^2 - 2\cos(\theta)\lambda + \lambda^2 \equiv 0$$

From here we can first simplify $1 = \cos(\theta)^2 + \sin(\theta)^2$ and then use the quadratic formula to attain the two possible λ values.

$$\lambda = \cos(\theta) \pm \sqrt{\cos(\theta)^2 - 1} = \cos(\theta) \pm i\sqrt{1 - \cos(\theta)^2} = \cos(\theta) \pm i\sqrt{\sin(\theta)^2}$$

In exponential phase notation we can write the two eigenvalues more concisely: $\lambda = e^{\pm i\theta}$

(e) Does the reflection matrix $\mathbf{T}$ across the x-axis in $\mathbb{R}^{2\times 2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Answer: Yes, both $+1$ and -1. Mechanically, we could go through the methods we have learned for attaining a characteristic polynomial from $\det(T - \lambda I) = (1-\lambda)(-1-\lambda) - (0)(0)$ and recalling our eigenvalues are the roots of this polynomial (the values where this polynomial is zero). This works because matrix $T - \lambda I$ only has a nonempty null space when its determinant is zero!

$$\det(T - \lambda I) = \lambda^2 - 1 \equiv 0 \quad \rightarrow \quad \lambda = \pm 1$$

Conceptually, we can reason that a vector along the x-axis will be unaffected by $\mathbf{T}$ (in this case $\lambda = +1$), where as a vector along the y-axis gets perfectly flipped by $\mathbf{T}$ (in this case $\lambda = -1$)

NOTE: A 2×2 reflection matrix always has $\lambda = \pm 1$, REGARDLESS of the axis of reflection. Why? Reflecting any vector that is on the reflection axis will not affect it (eigenvalue $+1$). Reflecting any vector orthogonal (perpendicular) to the reflection axis will just "flip it/negate it" (eigenvalue -1). In other words, the set of vectors that lie along the axis of reflection is the eigenspace associated with the eigenvalue $+1$ and the set of vectors orthogonal to the axis of reflection is the eigenspace associated with the eigenvalue -1.

(f) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

Answer: $N(A)$ is not just $\vec{0}$ as we have some $\vec{v} \neq \vec{0}$ satisfying $A\vec{v} = \lambda\vec{v}$. Another way we can state this is that $\dim(N(A)) > 0$.
Thus we can imagine if $\mathbf{M}\vec{x} = \vec{b}$ has a solution then $\mathbf{M}(\vec{x} + \vec{v}) = \vec{b}$ also solves the system, hence there are infinite solutions. Yet we also know that a nonzero null space means $\mathbf{M}$ has linearly dependent columns, so the vector $\vec{b}$ could lie outside of this span in which case there is no solution.

In summary, there are either infinite or no solutions to the system of equations $\mathbf{M}\vec{x} = \vec{b}$

(g) **(Practice)** Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Note that the matrix has linearly dependent columns. Therefore, according to part (f), one eigenvalue is $\lambda = 0$. The corresponding eigenvector, which is equivalent to the basis vector for the null space, is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The other eigenvalue is, by inspection, $\lambda = 1$ with the corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

