1. **(Optional) The Romulan Ruse** While scanning parts of the galaxy for alien civilization, the starship USS Enterprise NC-1701D encounters a Romulan starship that is known for advanced cloaking devices.

(a) **Concept: Matrix Transformations**

The Romulan illusion technology causes a point \((x_0, y_0)\) to transform or map to \((u_0, v_0)\). Similarly, \((x_1, y_1)\) is mapped to \((u_1, v_1)\). Figure 1 and Table 1 show two points on a Romulan ship and the corresponding mapped points.

![Figure 1: Figure for part (a)](image)

<table>
<thead>
<tr>
<th>Original Point</th>
<th>Mapped Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x_0, y_0) = (500, 500))</td>
<td>((u_0, v_0) = (500, 1500))</td>
</tr>
<tr>
<td>((x_1, y_1) = (1000, 500))</td>
<td>((u_1, v_1) = (1000, 1500))</td>
</tr>
</tbody>
</table>

Table 1: Original and Mapped Points

Find a transformation matrix \(A_0\) such that

\[
\begin{bmatrix}
    u_0 \\
    v_0
\end{bmatrix} = A_0 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ and } \begin{bmatrix}
    u_1 \\
    v_1
\end{bmatrix} = A_0 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.
\]

**Answer:** Let us assume \(A_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\). Hence for point \((x_0, y_0)\), we have:

\[
\begin{bmatrix}
    500 \\
    1500
\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 500 \\ 500 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\
    3
\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\
    1
\end{bmatrix}
\]

i.e.

\[a + b = 1; \quad (1)\]
\[c + d = 3. \quad (2)\]

Similarly, for point \((x_1, y_1)\), we have

\[
\begin{bmatrix}
    1000 \\
    1500
\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1000 \\ 500 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\
    3
\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\
    1
\end{bmatrix}
\]

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i.e.

\[ 2a + b = 2; \]  
\[ 2c + d = 3. \]

Solving Equations (1) and (3) for \( a \) and \( b \), we have:

\[ a = 1, \text{ and } b = 0. \]

Solving Equations (2) and (4) for \( c \) and \( d \), we have:

\[ c = 0, \text{ and } d = 3. \]

Substituting values of \( a, b, c, \) and \( d \), we have

\[ \mathbf{A}_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}. \]

Additionally, it can be observed from Figure 1 that the mapped vectors are derived by scaling the original vectors by 3 in the \( y \)-direction and by unity in the \( x \)-direction. Using Figure 1 and Table 1, we can write

\[ u_0 = x_0, \text{ and } v_0 = 3y_0, \]

and

\[ u_1 = x_1, \text{ and } v_1 = 3y_1. \]

Writing equations 5 and 6 in matrix-vector product form, we have

\[ \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}; \]

\[ \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}. \]

Hence

\[ \mathbf{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}. \]
(b) **Concept: Matrix Transformations**

In this scenario, every point on the Romulan ship \((x_m, y_m)\) is mapped to \((u_m, v_m)\), such that vector \[
\begin{bmatrix}
x_m \\
y_m
\end{bmatrix}
\]
is rotated counterclockwise by 30° and then scaled by 2 in the x- and y-directions. This transformation is shown in Figure 2.

\[
R_{\theta} = \begin{bmatrix}
\cos 30^\circ & -\sin 30^\circ \\
\sin 30^\circ & \cos 30^\circ
\end{bmatrix} = \begin{bmatrix}
\frac{\sqrt{3}}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{bmatrix}.
\]

Transformation matrix that rotates a vector counterclockwise by 30° and scales by 2 is:

\[
R = 2R_{\theta} = \begin{bmatrix}
\sqrt{3} & -1 \\
1 & \sqrt{3}
\end{bmatrix}.
\]

Alternatively, the transformation matrix can be written as:

\[
R = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix} \begin{bmatrix}
\cos 30^\circ & -\sin 30^\circ \\
\sin 30^\circ & \cos 30^\circ
\end{bmatrix} = \begin{bmatrix}
\sqrt{3} & -1 \\
1 & \sqrt{3}
\end{bmatrix}.
\]

Table 2: Trigonometric Table

<table>
<thead>
<tr>
<th>θ</th>
<th>sin θ</th>
<th>cos θ</th>
<th>tan θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>(\frac{1}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{1}{\sqrt{3}})</td>
</tr>
<tr>
<td>45°</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>1</td>
</tr>
<tr>
<td>60°</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{1}{2})</td>
<td>(\sqrt{3})</td>
</tr>
<tr>
<td>90°</td>
<td>1</td>
<td>0</td>
<td>(\infty)</td>
</tr>
</tbody>
</table>

Figure 2: Figure for part (b)
The Romulan ship has launched a probe into space and the Enterprise is trying to destroy the probe by firing a photon torpedo along a straight line from point \((0, 0)\) towards the probe.

(c) **Concept: Gaussian Elimination, Systems of Equations**
The Romulan generals found a clever way to hide the probe by transforming (mapping) its position with a *cloaking* (transformation) matrix \(A_p:\)

\[
A_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.
\]

They positioned the probe at \((x_p, y_p)\) so that it maps to \((u_p, v_p) = (0, 0)\), where \[
\begin{bmatrix} u_p \\ v_p \end{bmatrix} = A_p \begin{bmatrix} x_p \\ y_p \end{bmatrix}.
\]

This scenario is shown in Figure 3. The initial position of the torpedo is \((0, 0)\) and the torpedo cannot be fired on its initial position! Impressive trick indeed!

**Find the possible positions of the probe** \((x_p, y_p)\) **so that** \((u_p, v_p) = (0, 0)\).

**Answer:** We need to solve for

\[
\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

So essentially we need to find the nullspace of the matrix \(A_p\). Using Gaussian Elimination on the augmented matrix, we have:

\[
\begin{bmatrix} 1 & 3 & | & 0 \\ 2 & 6 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow x_p + 3y_p = 0 \Rightarrow x_p = -3y_p.
\]

The solution is \(\alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix}\), where \(\alpha \in \mathbb{R}\). So \(\begin{bmatrix} x_p \\ y_p \end{bmatrix}\) should be in the span of \(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\). Alternatively, any point \((x_p, y_p)\) that is on the line: \(x = -3y\), would represent all possible positions of the probe.
(d) **Concept: Eigenspaces/Eigenvectors/Eigenvalues**

It turns out the Romulan engineers were not as smart the Enterprise engineers. Their calculations did not work out and they positioned the probe at \((x_q, y_q)\) such that the *cloaking* (transformation) matrix, \(A_p\), mapped it to \((u_q, v_q)\), where

\[
\begin{bmatrix}
  u_q \\
  v_q
\end{bmatrix} = A_p \begin{bmatrix}
  x_q \\
  y_q
\end{bmatrix}, \text{ and } A_p = \begin{bmatrix}
  1 & 3 \\
  2 & 6
\end{bmatrix}.
\]

As a result, the torpedo while traveling along a straight line from \((0,0)\) to \((u_q, v_q)\), hit the probe at \((x_q, y_q)\) on the way!

The scenario is shown in Figure 4. For the torpedo to hit the probe, we must have \[
\begin{bmatrix}
  u_q \\
  v_q
\end{bmatrix} = \lambda \begin{bmatrix}
  x_q \\
  y_q
\end{bmatrix}, \text{ where } \lambda \text{ is a real number.}
\]

**Find the possible positions of the probe \((x_q, y_q)\) so that \((u_q, v_q) = (\lambda x_q, \lambda y_q)\). Remember that the torpedo cannot be fired on its initial position \((0,0)\).**

**Answer:** We need to solve for \(A_p \begin{bmatrix}
  x_q \\
  y_q
\end{bmatrix} = \lambda \begin{bmatrix}
  x_q \\
  y_q
\end{bmatrix}\), i.e. we need to find the eigenvectors of \(A_p\). Let’s start by finding the eigenvalues:

\[
\begin{align*}
\det \left( \begin{bmatrix}
  1 & 3 \\
  2 & 6
\end{bmatrix} - \lambda \begin{bmatrix}
  0 & 0 \\
  0 & \lambda
\end{bmatrix} \right) &= 0 \\
\det \left( \begin{bmatrix}
  1 - \lambda & 3 \\
  2 & 6 - \lambda
\end{bmatrix} \right) &= 0
\end{align*}
\]

So we have the characteristic polynomial:

\[(1 - \lambda)(6 - \lambda) - (3)(2) = 0 \Rightarrow \lambda = 0, 7\]

Using \(\lambda = 0\), we have: \[
\begin{bmatrix}
  1 & 3 \\
  2 & 6
\end{bmatrix} \begin{bmatrix}
  x_q \\
  y_q
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}, \text{ which will map } (x_q, y_q) \text{ to the original position of the torpedo. The torpedo cannot be fired on its original position. So } \lambda = 0 \text{ will not provide a valid solution.}
\]

Using \(\lambda = 7\), we have:

\[
(A_p - 7I) \begin{bmatrix}
  x_q \\
  y_q
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix} \Rightarrow \left( \begin{bmatrix}
  1 & 3 \\
  2 & 6
\end{bmatrix} - 7 \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix} \right) \begin{bmatrix}
  x_q \\
  y_q
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix} \Rightarrow \begin{bmatrix}
  -6 & 3 \\
  2 & -1
\end{bmatrix} \begin{bmatrix}
  x_q \\
  y_q
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

Using Gaussian Elimination on the augmented matrix form, we have

\[
\begin{bmatrix}
  -6 & 3 & 0 \\
  2 & -1 & 0
\end{bmatrix} \Rightarrow \begin{bmatrix}
  2 & -1 & 0 \\
  0 & 0 & 0
\end{bmatrix} \Rightarrow 2x_q - y_q = 0 \Rightarrow y_q = 2x_q
\]

The solution is \(x \begin{bmatrix}
  1 \\
  2
\end{bmatrix}\), where \(x \in \mathbb{R} : x \neq 0\). So \(\begin{bmatrix}
  x_q \\
  y_q
\end{bmatrix}\) should be in the span of \(\begin{bmatrix}
  1 \\
  2
\end{bmatrix}\).

Alternatively, any point \((x_q, y_q)\) that is on the line: \(y = 2x\), excluding \((0,0)\), would represent all possible positions of the probe.
2. (Optional) Proof

*Concept: Null Spaces, Invertibility*

Consider a square matrix $A$. Prove that if $A$ has a non-trivial nullspace, i.e. if the nullspace of $A$ contains more than just $\vec{0}$, then matrix $A$ is not invertible.

**Answer:** We are given that the nullspace of $A$ contains a vector other than $\vec{0}$. Let such a vector be $\vec{y} \neq \vec{0}$, where $A\vec{y} = \vec{0}$. Imagine, for the sake of contradiction, that $A$ had an inverse $A^{-1}$. Then we find that

$$A\vec{y} = \vec{0}$$

$$\implies (A^{-1}A)\vec{y} = A^{-1}\vec{0}$$

$$\implies \vec{y} = \vec{0},$$

since by the definition of an inverse, $A^{-1}A = I$.

But we said that $\vec{y} \neq \vec{0}$, so this is a contradiction! Therefore, our original hypothesis must have been false, so $A$ cannot have an inverse.

Thus, the matrix $A$ is not invertible.