1. Passive Sign Convention and Power

(a) We have made four copies of a circuit below. Following passive sign convention, there are four different possible labelings of current directions and voltage polarities for the circuit. For each copy, label each circuit’s voltage source and resistor with current direction and voltage polarity labelings, keeping with passive sign convention.

\[ V_s \quad + \quad R_1 \quad + \quad V_s \quad - \quad R_1 \quad + \quad V_s \quad - \quad R_1 \quad + \quad V_s \quad - \quad R_1 \]

Answer:

\[ V_s \quad + \quad V_1 \quad - \quad i_1 \quad V_s \quad + \quad V_1 \quad - \quad i_1 \quad V_s \quad - \quad V_1 \quad + \quad i_2 \quad V_s \quad - \quad V_1 \quad + \quad i_2 \]

(b) Suppose we consider one of the possible labelings you have found above. Calculate the power dissipated or supplied by every element in the circuit. Let \( V_s = 5 \text{V} \) and let \( R_1 = 5 \text{\Omega} \).

Answer: We’ll start by solving the circuit for the unknown node potentials and currents.
The KCL equation for the one node in this circuit is:

\[ i_1 + i_2 = 0 \]

The element equations for the two elements in this circuit are:

\[ u_1 - 0 = V_1 = V_s \]
\[ u_1 - 0 = V_{R_1} = i_2 R_1 \]

Solving the above equations with \( V_s = 5 \text{ V} \) and \( R_1 = 5 \Omega \):

\[ u_1 = 5 \text{ V} \]
\[ i_1 = -1 \text{ A} \]
\[ i_2 = 1 \text{ A} \]

From above, we can solve for the power dissipated across the resistor:

\[ P_{R_1} = i_2 V_{R_1} = 1 \text{ A} \cdot 5 \text{ V} = 5 \text{ W} \]

Next we can solve for the power dissipated across the voltage source:

\[ P_{V_s} = i_1 V_1 = i_1 V_s = -1 \text{ A} \cdot 5 \text{ V} = -5 \text{ W} \]

Notice we calculate a negative value for the power dissipated by the voltage source, implying the voltage source is adding power to the circuit.

(c) Suppose we choose a second labeling of the circuit as shown below. Calculate the power dissipated or supplied by every element in the circuit. Let \( V_s = 5 \text{ V} \) and let \( R_1 = 5 \Omega \).

\[-V_s + \]
\[ + \]
\[ V_1 \]
\[ i_1 + \]
\[ - \]
\[ R_1 \]
\[ V_{R_1} \]
\[ i_2 \]

**Answer:** We’ll solve the circuit the same way as last time.
The KCL equation for the one node in this circuit is:

\[-i_1 + i_2 = 0\]

The element equations for the two elements in this circuit are:

\[0 - u_1 = V_1 = -V_s\]
\[u_1 - 0 = V_{R_1} = i_2 R_1\]

Solving the above equations with \(V_s = 5\) V and \(R_1 = 5\) Ω:

\[u_1 = 5\] V
\[i_1 = 1\] A
\[i_2 = 1\] A

From above, we can solve for the power dissipated across the resistor:

\[P_{R_1} = i_2 V_{R_1} = 1\text{ A} \cdot 5\text{ V} = 5\text{ W}\]

Next we can solve for the power dissipated across the voltage source:

\[P_{V_s} = i_1 V_1 = i_1 (-V_s) = 1\text{ A} \cdot -5\text{ V} = -5\text{ W}\]

Notice here that the circuit has the same power dissipated by all the elements. This is because with both labeling of currents, we followed the passive sign convention.

(d) Did the values of the element voltages and element currents change with the different labeling? Did the power for each circuit element change? Did the node voltages change? If a quantity didn’t change with a difference in labeling, discuss what would have to change for quantity to change.

**Answer:** With a different labeling, element voltages and element currents will change. The quantities were \(V_i = 5\) V and \(i_i = -1\) A in (b) and in (c) \(V_i = -5\) V and \(i_i = 1\) A. Flipping the direction of a labeled current or the polarity of a labeled voltage will lead to negation of the value.

The power dissipated by a circuit element will not change because we follow passive sign convention. Passive sign convention requires that if we flip the direction of an element current we also flip the polarity of the corresponding element voltage, so there is a double negation in the computation of power. The only way to get a different value of power would be to change the component values or the circuit diagram itself by removing or adding more circuit elements. A physical system will only have one behavior as governed by the laws of physics - how we compute our answer should not change how it behaves. Our labeled voltage polarities and current directions are more akin to measurement choices which can change what we see.

The node voltages too, did not change. The top node voltage in both labelings were 5 V and the bottom node voltages were 0 V. What would have to change to alter these values is one of three things: either the location of ground, the circuit component values, or the circuit diagram itself.
2. Circuit Analysis

Provided the circuit below...

(a) ...use nodal analysis to solve for all node voltages.
(b) ...find the current $I_{R_3}$ flowing through resistor $R_3$.

$$\begin{align*}
I_S &\quad \text{at node 1} \\
R_1 &\quad R_2 &\quad R_3 &\quad R_4 \\
V_S &\quad \text{at node 0} \\
\end{align*}$$

Answer:
Again it is a splendid idea to implement our circuit analysis procedure outlined in class!

**Step 1** Selecting a ground node; this has already been done for us.

**Step 2** Label all known voltages determined by voltage sources (denoted below as $u_1, u_4$).

- We know $u_4 = 0$ since this node is directly linked to ground.
- Also we know $u_1 = +V_s$ because the element linking node $u_1$ to ground is the voltage source $V_s$ AND that source has the negative lead facing the ground side.

**Step 3** Label all remaining unknown nodes (denoted below as $u_2, u_3$).

**Step 4** Label element voltages and currents ($V_{R_1}, I_{R_1}$, etc.) .

$$\begin{align*}
V_{S_1} &\quad \text{at node 1} \\
R_1 &\quad R_2 &\quad R_3 &\quad R_4 \\
V_S &\quad \text{at node 0} \\
\end{align*}$$
Note: You can pick any direction you want for the currents, but once you have done so the voltages must abide by the passive sign convention!! Vice versa, you can pick the element voltage sign first but then the current has to abide by the passive sign convention.

**Step 5** Write KCL equations for all nodes with unknown voltages (namely $u_2,u_3$):

\[
I_{R_2} + I_{S_1} - I_{R_1} = 0 \\
I_{R_3} - I_{S_1} - I_{R_4} = 0
\]

**Step 6 / 7** Find expressions for all element currents in terms of element voltages and characteristics.

Step 7 is where we substitute in the node voltage expressions for known quantities ($u_1 = +V_s, u_4 = 0$).

\[
I_{R_1}R_1 = u_2 - u_4 = u_2 \\
I_{R_2}R_2 = u_1 - u_2 = V_s - u_2 \\
I_{R_3}R_3 = u_1 - u_3 = V_s - u_3 \\
I_{R_4}R_4 = u_3 - u_4 = u_3
\]

**Step 8** Substitute expressions found in step 7 into the KCL equations from step 5.

\[
\frac{V_s - u_2}{R_2} + \frac{u_2}{R_1} = 0 \\
\frac{V_s - u_3}{R_3} - \frac{u_3}{R_4} = 0
\]

**Step 9** Notice that we can now directly solve for $u_2$ and $u_3$ from the results of step 8.

\[
0 = \frac{V_s}{R_2} + I_s - u_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \quad \rightarrow \quad u_2 = \frac{V_sR_1 + I_sR_1R_2}{R_1 + R_2} \tag{a}
\]

\[
0 = \frac{V_s}{R_3} - I_s - u_3 \left( \frac{1}{R_4} - \frac{1}{R_3} \right) \quad \rightarrow \quad u_3 = \frac{V_sR_4 - I_sR_4R_3}{R_4 + R_3} \tag{a}
\]

Then we can substitute into the equations of Step 6 to find the currents through each resistor. This finally yields current $I_{R_3}$.

\[
I_{R_3} = \frac{V_s - u_3}{R_3} = \frac{V_s - \frac{V_sR_4 - I_sR_4R_3}{R_3}}{R_3} = \frac{V_s + I_sR_4}{R_3 + R_4} \tag{b}
\]

3. **(PRACTICE) Current Divider**

So far we’ve shown that for the circuit below to the left, the voltage across the resistor $R_2$ is expressed

\[
V_{R_2} = \left( \frac{R_2}{R_1 + R_2} \right) V_s.
\]

Let us derive a similar formula for the right circuit below, relating the current $I_{R_2}$ through $R_2$ to the current through the current source $I_s$.

![Current Divider Circuit](image)
Answer:
It would be an excellent idea to run through the circuit analysis procedure outlined in class:

**Step 1** Selecting a ground node; this has already been done for us.

**Step 2** Label all known voltages determined by voltage sources (there are none in this circuit).

**Step 3** Label all remaining unknown nodes (denoted below $u_1$).

**Step 4** Label element voltages and currents ($V_{R1}, I_{R1}$, etc.)

![Diagram of circuit]

Note: You can pick any direction you want for the currents, but once you have done so the voltages must abide by the passive sign convention!! Vice versa, you can pick the element voltage sign first but then the current has to abide by the passive sign convention.

**Step 5** Write KCL equations for all nodes with unknown voltages (namely $u_1$)

\[ I_s - I_{R1} - I_{R2} = 0 \]

where we used the fact that $i_s = I_s$.

**Step 6** Find expressions for all element currents in terms of element voltages and characteristics

\[ V_{R1} = I_{R1}R_1 \quad V_{R2} = I_{R2}R_2. \]

**Step 7** Replace each element voltage expression using our node voltage terms (only $u_1$ for this circuit)

\[ (u_1 - 0) = I_{R1}R_1 \quad (u_1 - 0) = I_{R2}R_2. \]

**Step 8** Plug our expressions found in step 7 into the KCL equations from step 5

\[ I_s - \frac{u_1}{R_1} - \frac{u_1}{R_2} = 0 \quad \rightarrow \quad u_1 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = I_s \quad \rightarrow \quad u_1 \left( \frac{R_2}{R_1R_2} + \frac{R_1}{R_2R_1} \right) = I_s \quad \rightarrow \quad u_1 \left( \frac{R_2 + R_1}{R_1R_2} \right) = I_s \]

**Step 9** With the node voltage $u_1$ known explicitly now, replug back into our expressions from step 7 to finally identify the current passing through each resistor!

\[ u_1 = \frac{R_1R_2}{R_1 + R_2}I_s \]

From step 7 we plug in to find $I_{R1}$ and $I_{R2}$ from known circuit quantities

\[ I_{R1} = \frac{u_1}{R_1} \quad \rightarrow \quad I_{R1} = \frac{R_2}{R_1 + R_2}I_s \]

\[ I_{R2} = \frac{u_2}{R_2} \quad \rightarrow \quad I_{R2} = \frac{R_1}{R_1 + R_2}I_s \quad \square \]