1. Series and Parallel Combinations

For the resistor network shown below, find an equivalent resistance between the terminals $A$ and $B$ using the resistor combination rules for series and parallel resistors.

\[ 5\,\text{k}\Omega \parallel ((6\,\text{k}\Omega \parallel 6\,\text{k}\Omega + 8\,\text{k}\Omega) = 5\,\text{k}\Omega \parallel (2\,\text{k}\Omega + 8\,\text{k}\Omega) = 5\,\text{k}\Omega \parallel 10\,\text{k}\Omega = 3.33\,\text{k}\Omega) \]

Answer:
2. Superposition

For the following circuits:

i. Use the superposition theorem to solve for the voltages across the resistors.

ii. For parts (b) and (c) only, find the power dissipated/generated by all components. Is power conserved?

(a)

Answer: Turning on only $V_{S_1}$, we have the following voltages across the resistors:

$$V_{R_1} = \frac{R_1}{R_1 + R_2} V_{S_1}$$
$$V_{R_2} = \frac{R_2}{R_1 + R_2} V_{S_1}$$
$$V_{R_3} = 0$$

Then turning only $I_{S_1}$, we have the following voltages:

$$V_{R_1} = \frac{R_1 R_2}{R_1 + R_2} I_{S_1}$$
$$V_{R_2} = -\frac{R_1 R_2}{R_1 + R_2} I_{S_1}$$
$$V_{R_3} = I_{S_1} R_3$$

Using superposition we can sum up the contributions from both $V_{S_1}$ and $I_{S_1}$ to get:

$$V_{R_1} = \frac{R_1}{R_1 + R_2} V_{S_1} + \frac{R_1 R_2}{R_1 + R_2} I_{S_1}$$
$$V_{R_2} = V_{S_1} - V_{R_1} = \frac{R_2}{R_1 + R_2} V_{S_1} - \frac{R_1 R_2}{R_1 + R_2} I_{S_1}$$
$$V_{R_3} = I_{S_1} R_3$$

(b)
Answer:
i. While we could apply the algorithm we have learned in class, let’s see if there’s a way to find the answer quicker than before. We’re looking for the voltage across the resistor, which could be found quickly using Ohm’s law if we knew the current. If we were to apply KCL at the node at the top of the circuit, one source is coming in, the other source is leaving, and the current through the resistor is leaving. From KCL, we then know $i_{R_1} = I_{S_1} - I_{S_2}$. Applying Ohm’s Law we find:

\[ V_{R_1} = (I_{S_1} - I_{S_2})R_1 \]

We could also solve this using superposition. Turning on $I_{S_1}$ gives $V_{R_1} = I_{S_1}R_1$. Turning on $I_{S_2}$ gives $V_{R_1} = -I_{S_2}R_1$. Finally, the total $V_{R_1}$ is the sum of the individual $V_{R_1}$’s or

\[ V_{R_1} = (I_{S_1} - I_{S_2})R_1 \]

ii. Power is conserved.

(c) (PRACTICE)

Answer:
i. Once again, we could apply the circuit analysis algorithm or find the answer directly. Notice the circuit only has one loop, so we can use KVL to find the voltage across the resistor.

\[ V_{R_1} = V_{S_1} - V_{S_2} \]

We could also solve with superposition. Turning on $V_{S_1}$ gives $V_{R_1} = V_{S_1}$. Turning on $V_{S_2}$ gives $V_{R_1} = -V_{S_2}$. The overall voltage is then the sum.

\[ V_{R_1} = V_{S_1} - V_{S_2} \]

ii. Power is conserved.

\[ P_{R_1} = \frac{V_{R_1}^2}{R_1} = \frac{(V_{S_1} - V_{S_2})^2}{R_1} \]

\[ P_{V_{S_1}} = -V_{R_1}V_{S_1} = -\frac{V_{S_1}(V_{S_1} - V_{S_2})}{R_1} \]
\[ P_{V_2} = I_{R_1} V_{S_2} = \frac{V_{S_2} (V_{S_1} - V_{S_2})}{R_1} \]

\[ P_{R_1} + P_{V_{S_1}} + P_{V_{S_2}} = \frac{(V_{S_1} - V_{S_2})^2}{R_1} - \frac{V_{S_1} (V_{S_1} - V_{S_2})}{R_1} + \frac{V_{S_2} (V_{S_1} - V_{S_2})}{R_1} = 0 \]

Power is conserved.
3. Current Sources And Capacitors

Given the circuit below, find an expression for $v_{\text{out}}(t)$ in terms of $I_s$, $C$, $V_0$, and $t$, where $V_0$ is the initial voltage across the capacitor at $t = 0$.

![Circuit Diagram]

Then plot the function $v_{\text{out}}(t)$ over time on the graph below for the following conditions detailed below. Use the values $I_s = 1 \text{mA}$ and $C = 2 \mu\text{F}$.

(a) Capacitor is initially uncharged $V_0 = 0$ at $t = 0$.
(b) Capacitor has been charged with $V_0 = +1.5\text{V}$ at $t = 0$.
(c) Practice: Swap this capacitor for one with half the capacitance $C = 1 \mu\text{F}$, which is initially uncharged $V_0 = 0$ at $t = 0$.

HINT: Recall the calculus identity $\int_a^b f'(x)\,dx = f(b) - f(a)$, where $f'(x) = \frac{df}{dx}$.

Answer:
The key here is to exploit the capacitor equation by taking its time-derivative

$$Q = C V_{\text{out}} \implies \frac{dQ}{dt} = I_s = C \frac{dV_{\text{out}}}{dt}.$$

From here we can rearrange and show that $\frac{dV_{\text{out}}}{dt} = \frac{I_s}{C}$.
Thus the voltage has a constant slope!
Our solution is

$$V_{\text{out}}(t) = V_0 + \left(\frac{I_s}{C}\right) t \quad \square$$
To be more mathematically formal, we are solving a differential equation that happens to return a linear function for $v_{\text{out}}(t)$:

$$\frac{dV_{\text{out}}}{dt} = \frac{I_s}{C} \quad \rightarrow \quad \int_0^t \frac{dV_{\text{out}}}{dt} \, dt = V_{\text{out}}(t) - V_{\text{out}}(0) = \int_0^t \frac{I_s}{C} \, dt \equiv \frac{I_s}{C} t$$

Thus we arrive at the same statement as seen earlier $V_{\text{out}}(t) = V_{\text{out}}(0) + \left( \frac{I_s}{C} \right) t$.

For all parts, we have $I_s = 1\, \text{mA}$. For part (a), we have $V_{\text{out}}(0) = 0\, \text{V}$ and $C = 2\, \mu\text{F}$. Plugging this into our equation, we get:

$$V_{\text{out}}(t) = 0\, \text{V} + \frac{1\, \text{mA}}{2\, \mu\text{F}} = \left( \frac{1 \, \text{V}}{2 \, \text{ms}} \right) t$$

For part (b), this only changes by an intercept / initial condition $V_{\text{out}}(0) = 1.5\, \text{V}$:

$$V_{\text{out}}(t) = 1.5\, \text{V} + \frac{1\, \text{mA}}{2\, \mu\text{F}} = 1.5\, \text{V} + \left( \frac{1 \, \text{V}}{2 \, \text{ms}} \right) t$$

For part (c), we have $V_{\text{out}}(0) = 0\, \text{V}$ and $C = 1\, \mu\text{F}$.

$$V_{\text{out}}(t) = 0\, \text{V} + \frac{1\, \text{mA}}{1\, \mu\text{F}} = \left( \frac{1 \, \text{V}}{\text{ms}} \right) t$$

With half the capacitance and the same current, we get that the slope is twice as large. Some physical intuition: capacitors are like buckets, and charge is like the total water. Voltage is a measure of the height of the water in the bucket. Capacitance is the area of the opening of the bucket, i.e. the area of top hole or the lid. If the opening, like the capacitance, is halved, and a hose filling the bucket, like a current source, has constant flow, then the height of the water will grow twice as fast.

When plotting, make sure to recall (a) and (c) start at the origin, while (b) has initially charged plates by $V_0 = 1.5\, \text{V}$. Results are shown below.