EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 11B

Reference: Inner products

For this course we will use a standard inner product definition from matrix-vector multiplication:

$$\langle \vec{x}, \vec{y} \rangle \;=\; x_1 y_1 + x_2 y_2 + \ldots + x_n y_v \;, \quad \text{for any } \; \vec{x}, \vec{y} \in \mathbb{R}^n.$$

In general, any inner product $\langle \cdot, \cdot \rangle$ on a real vector space $\mathbb{V}$ is a bilinear function that satisfies the following three properties:

(a) **Symmetry:** $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

(b) **Linearity:** $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$ and $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$, where $c \in \mathbb{R}$ is a real number.

(c) **Non-negativity:** $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$.

Here $\vec{x}$, $\vec{y}$, and $\vec{z}$ can be any vectors in the vector space $\mathbb{V}$.
The norm (or length) of a vector $\vec{x} = [x_1, x_2, ..., x_n]^T$ is defined using the inner product as

$$\|\vec{x}\| \;=\; \sqrt{\langle \vec{x}, \vec{x} \rangle} \;\equiv\; \sqrt{x_1^2 + x_2^2 + ... + x_n^2}\;.$$

1. Inner Product Properties

For this question we will verify our coordinate definition of the inner product

$$\langle \vec{x}, \vec{y} \rangle \;=\; x_1 y_1 + x_2 y_2 + \ldots + x_n y_v \;, \quad \text{for any } \; \vec{x}, \vec{y} \in \mathbb{R}^n$$

indeed satisfies the key properties required for all inner products, but presently for the 2-dimensional case. Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ for the following parts:

(a) Show symmetry $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$:

Answer: This is seen by direct expansion:
Let $x_i, y_i \in \mathbb{R}$, then

$$\begin{aligned}\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle &= x_1 \cdot y_1 + x_2 \cdot y_2 \\ &= y_1 \cdot x_1 + y_2 \cdot x_2 \\ &= \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle\end{aligned}$$

(b) Show linearity $\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle$, where $c \in \mathbb{R}$ is a real number.

Answer: This is accomplished through a direct expansion:

$$\begin{aligned}\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, c\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + d\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} cy_1 + dz_1 \\ cy_2 + dz_2 \end{bmatrix} \right\rangle \\ &= x_1(cy_1 + dz_1) + x_2(cy_2 + dz_2) \\ &= c(x_1y_1 + x_2y_2) + d(x_1z_1 + x_2z_2) \\ &= c\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle + d\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle \\ &= c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle\end{aligned}$$

(c) Show non-negativity $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$:

Answer: This part requires just a bit more thought beyond a direct expansion of $\langle \vec{x}, \vec{x} \rangle$, but we first recognize that this inner product is the definition of the norm (or length) of $\vec{x}$. So it is at least in intuitive that a length of some vector (squared) cannot be negative:

$$\begin{aligned}\langle \vec{x}, \vec{x} \rangle &= \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \\ &= x_1^2 + x_2^2\end{aligned}$$

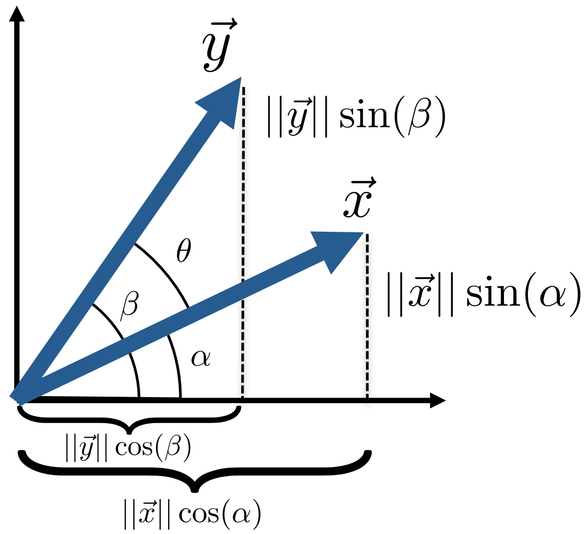
From this result we notice if either x_1 or x_2 are nonzero (even negative) values, then the inner product HAS to be positive. The only case in which the inner product $\langle \vec{x}, \vec{x} \rangle$ is identically zero is when both $x_1 = 0$ AND $x_2 = 0$, which verifies the final part of the property: $\langle \vec{x}, \vec{x} \rangle = 0$ ONLY IF $\vec{x} = \vec{0}$.

As a bonus, suppose we re-label our vector components $x_1 = a$ and $x_2 = b$.
The we see $\langle \vec{x}, \vec{x} \rangle = c^2 = a^2 + b^2$, which is the Pythagorean theorem!
This verifies that $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = c$ can be geometrically understood as the length of vector $\vec{x}$.

2. Geometric Interpretation of the Inner Product

In this problem we explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in $\mathbb{R}^2$.

(a) Derive a formula for the inner product of two vectors in terms of their magnitudes and the angle between them. The figure below may be helpful:



Answer: From trigonometric calculation, if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then we know that $x_1 = \|\vec{x}\| \cdot \cos\alpha$, $x_2 = \|\vec{x}\| \cdot \sin\alpha$, $y_1 = \|\vec{y}\| \cdot \cos\beta$ and $y_2 = \|\vec{y}\| \cdot \sin\beta$ (as in the figure). Then you can directly write

$$
\begin{aligned}
\langle \vec{x}, \vec{y} \rangle &= x_1 \cdot y_1 + x_2 \cdot y_2 = \\
&= \underbrace{\|\vec{x}\| \cdot \cos\alpha}_{x_1} \cdot \underbrace{\|\vec{y}\| \cdot \cos\beta}_{y_1} + \underbrace{\|\vec{x}\| \cdot \sin\alpha}_{x_2} \cdot \underbrace{\|\vec{y}\| \cdot \sin\beta}_{y_2} \\
&= \|\vec{x}\| \|\vec{y}\| \left(\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta\right) = \\
&= \|\vec{x}\| \|\vec{y}\| \cdot \cos\left(\beta - \alpha\right) \\
&= \|\vec{x}\| \|\vec{y}\| \cdot \cos\theta
\end{aligned}
$$

(b) For each sub-part, identify any two (nonzero) vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ that satisfy the stated condition and compute their inner product.

i. Identify a pair of parallel vectors:

Answer: Parallel vectors point in the same direction (have an angle of 0° between them).
This means we must have $\vec{y} = \alpha\vec{x}$ for some $\alpha > 0$.
Having only this condition leaves a lot of freedom.
Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = 2\,\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot 2 + 1 \cdot 2 = 4$$

ii. Identify a pair of anti-parallel vectors:

Answer: Anti-parallel vectors point in opposite directions (have an angle of 180° between them).
This means we must have $\vec{y} = \alpha\vec{x}$ again, but now for some negative $\alpha < 0$.
Having only this condition still leaves a lot of freedom.

Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and then set $\vec{y} = -2\,\vec{x} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot -2 + 0 \cdot 0 = -2$$

iii. Identify a pair of perpendicular vectors:

Answer: Anti-parallel vectors point in 90° directions with respect to each-other.
Most importantly, the Euclidean inner product $\langle \vec{x}, \vec{y} \rangle = 0$ whenever $\vec{x}, \vec{y}$ are perpendicular.

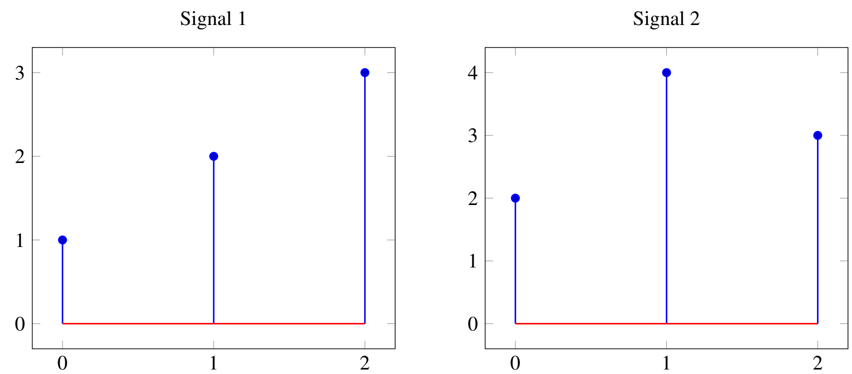
For our example we will fix $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and then leave $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ general.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot y_1 + 0 \cdot y_2 = y_1 \ \equiv 0.$$

Thus we must set $y_1 = 0$, but y_2 can assume any nonzero value!

3. Correlation

We are given the following two signals, $s_1[n]$ and $s_2[n]$ respectively.



Find the cross correlations, $\text{corr}_{s_1}(s_2)$ and $\text{corr}_{s_2}(s_1)$ for signals $s_[n]$ and $s_2[n]$. Recall

$$\text{corr}_x(y)[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k].$$

$$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$$

$\vec{s}_1$	0	0	1	2	3	0	0
$\vec{s}_2[n+2]$							
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	+	+	+	+	+	+	=

$\vec{s}_1$	0	0	1	2	3	0	0
$\vec{s}_2[n+1]$							
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	+	+	+	+	+	+	=

$\vec{s}_1$	0	0	1	2	3	0	0
$\vec{s}_2[n]$							
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	+	+	+	+	+	+	=

$\vec{s}_1$	0	0	1	2	3	0	0
$\vec{s}_2[n-1]$							
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	+	+	+	+	+	+	=

$\vec{s}_1$	0	0	1	2	3	0	0
$\vec{s}_2[n-2]$							
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	+	+	+	+	+	+	=

$$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$$

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n+2]$														
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$		+		+		+		+		+		+		=

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n+1]$														
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$		+		+		+		+		+		+		=

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n]$														
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$		+		+		+		+		+		+		=

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n-1]$														
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$		+		+		+		+		+		+		=

$\vec{s}_2$	0		0		2		4		3		0		0	
$\vec{s}_1[n-2]$														
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$		+		+		+		+		+		+		=

Answer: The linear cross-correlation is calculated by shifting the second signal both forward and backward until there is no overlap between the signals. When there is no overlap, the cross-correlation goes to zero. Both of these cross-correlations should have only zeros outside the range: $-2 \leq n \leq 2$.

$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+2]$	2		4		3		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	0	+	0	+	3	+	0	+	0	+	0	+	0	$=3$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+1]$	0		2		4		3		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	0	+	0	+	4	+	6	+	0	+	0	+	0	$=10$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n]$	0		0		2		4		3		0		0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	0	+	0	+	2	+	8	+	9	+	0	+	0	$=19$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n-1]$	0		0		0		2		4		3		0	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0	+	0	+	0	+	4	+	12	+	0	+	0	$=16$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n-2]$	0	0	0	0	2	4	3	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0 +	0 +	0 +	0 +	6 +	0 +	0	$= 6$

$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$

$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\vec{s}_1[n+2]$	1	2	3	0	0	0	0	
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$	0 +	0 +	6 +	0 +	0 +	0 +	0	$= 6$

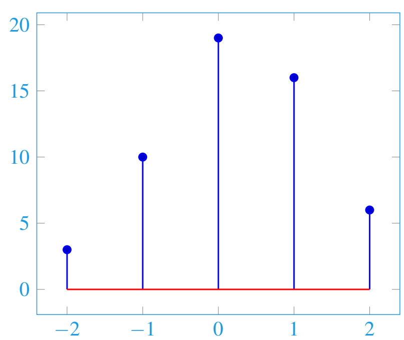
$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\vec{s}_1[n+1]$	0	1	2	3	0	0	0	
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$	0 +	0 +	4 +	12 +	0 +	0 +	0	$= 16$

$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\vec{s}_1[n]$	0	0	1	2	3	0	0	
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	0 +	0 +	2 +	8 +	9 +	0 +	0	$= 19$

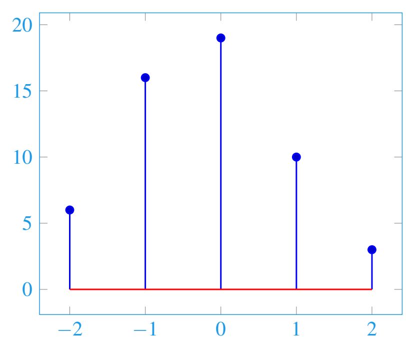
$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\vec{s}_2[n-1]$	0	0	0	1	2	3	0	
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	0 +	0 +	0 +	4 +	6 +	0 +	0	$= 10$

$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\vec{s}_2[n-2]$	0	0	0	0	1	2	3	
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	0 +	0 +	0 +	0 +	3 +	0 +	0	$= 3$

Cross-correlation of $\vec{s}_1$ and $\vec{s}_2$



Cross-correlation of $\vec{s}_2$ and $\vec{s}_1$



Notice that $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k] = \text{corr}_{\vec{s}_2}(\vec{s}_1)[-k]$, i.e. changing the order of the signals reverses the cross-correlation sequence.

