1. Inner Product Properties

For this question, we will verify our coordinate definition of the inner product

\[ \langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n, \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n \]

indeed satisfies the key properties required for all inner products, but presently for the 2-dimensional case.

Suppose \( \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2 \) for the following parts:

(a) Show symmetry \( \langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle \).

**Answer:** This is seen by direct expansion:

Let \( x_i, y_i \in \mathbb{R} \), then

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 \cdot y_1 + x_2 \cdot y_2 \\
= y_1 \cdot x_1 + y_2 \cdot x_2 \\
= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

(b) Show linearity \( \langle \vec{x}, c \vec{y} + d \vec{z} \rangle = c \langle \vec{x}, \vec{y} \rangle + d \langle \vec{x}, \vec{z} \rangle \), where \( c, d \in \mathbb{R} \) are real numbers.

**Answer:** This is accomplished through a direct expansion:

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} c y_1 \\ c y_2 \end{bmatrix} + d \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} cy_1 + dz_1 \\ cy_2 + dz_2 \end{bmatrix} \\
= x_1 (cy_1 + dz_1) + x_2 (cy_2 + dz_2) \\
= c (x_1 y_1 + x_2 y_2) + d (x_1 z_1 + x_2 z_2) \\
= c \langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rangle + d \langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \rangle \\
= c \langle \vec{x}, \vec{y} \rangle + d \langle \vec{x}, \vec{z} \rangle
\]
(c) Show non-negativity \( \langle \vec{x}, \vec{x} \rangle \geq 0 \), with equality if and only if \( \vec{x} = \vec{0} \).

**Answer:** This part requires just a bit more thought beyond a direct expansion of \( \langle \vec{x}, \vec{x} \rangle \), but we first recognize that this inner product is the definition of the norm (or length) of \( \vec{x} \). So it is at least in intuitive that a length of some vector (squared) cannot be negative:

\[
\langle \vec{x}, \vec{x} \rangle = \langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rangle = x_1^2 + x_2^2
\]

From this result we notice if either \( x_1 \) or \( x_2 \) are nonzero (even negative) values, then the inner product HAS to be positive. The only case in which the inner product \( \langle \vec{x}, \vec{x} \rangle \) is identically zero is when both \( x_1 = 0 \) AND \( x_2 = 0 \), which verifies the final part of the property: \( \langle \vec{x}, \vec{x} \rangle = 0 \) ONLY IF \( \vec{x} = \vec{0} \).

As a bonus, suppose we re-label our vector components \( x_1 = a \) and \( x_2 = b \).

The we see \( \langle \vec{x}, \vec{x} \rangle = c^2 = a^2 + b^2 \), which is the Pythagorean theorem!

This verifies that \( \| \vec{x} \| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = c \) can be geometrically understood as the length of vector \( \vec{x} \).

2. **Geometric Interpretation of the Inner Product**

In this problem, we explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in \( \mathbb{R}^2 \).

Remember that the formula for the inner product of two vectors can be expressed in terms of their magnitudes and the angle between them as follows:

\[
\langle \vec{x}, \vec{y} \rangle = \| \vec{x} \| \| \vec{y} \| \cos \theta
\]

The figure below may be helpful in illustrating this property:

![Geometric Interpretation of the Inner Product](image)

For each sub-part, give an example of any two (nonzero) vectors \( \vec{x}, \vec{y} \in \mathbb{R}^2 \) that satisfy the stated condition and compute their inner product.

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(a) Give an example of a pair of parallel vectors (vectors that point in the same direction and have an angle of 0 degrees between them).

**Answer:** Parallel vectors point in the same direction (have an angle of $0^\circ$ between them).
This means we must have $\vec{y} = \alpha \vec{x}$ for some $\alpha > 0$.
Having only this condition leaves a lot of freedom.
Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = 2 \vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot 2 + 1 \cdot 2 = 4$$

(b) Give an example of a pair of anti-parallel vectors (vectors that point in opposite directions).

**Answer:** Anti-parallel vectors point in opposite directions (have an angle of $180^\circ$ between them).
This means we must have $\vec{y} = \alpha \vec{x}$ again, but now for some negative $\alpha < 0$.
Having only this condition still leaves a lot of freedom.
Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and then set $\vec{y} = -2 \vec{x} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot -2 + 1 \cdot -2 = -4$$

(c) Give an example of a pair of perpendicular vectors (vectors that have an angle of 90 degrees between them).

**Answer:** Perpendicular vectors point in $90^\circ$ directions with respect to each-other.
Most importantly, the Euclidean inner product $\langle \vec{x}, \vec{y} \rangle = 0$ whenever $\vec{x}, \vec{y}$ are perpendicular.

For our example we will fix $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and then leave $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ general.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot y_1 + 0 \cdot y_2 = y_1 \equiv 0.$$  

Thus we must set $y_1 = 0$, but $y_2$ can assume any nonzero value!

3. **Correlation**

(a) You are given the following two signals:
Sketch the linear cross-correlation of signal 1 with signal 2, that is find: $\text{corr}(\vec{s}_1, \vec{s}_2)[n]$ for $n = 0, 1, \ldots, 4$. Do not assume the signals are periodic.

**Answer:**

Represent signal 1 as the vector $\vec{s}_1 = [4 \ 2 \ 0 \ 0 \ -2 \ 0 \ 0 \ 0 \ 0]^T$, zero-padded so that we compute only the linear correlation. Similarly, represent signal 2 as the vector $\vec{s}_2 = [-4 \ 8 \ -4 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$, where we once again zero pad the vector. Notice that we zero pad the vectors $\vec{s}_1$ and $\vec{s}_2$ to represent the signals from $n = 0, 1, \ldots, 8$. This is because we are only interested in calculating the cross-correlation for for $n = 0, 1, \ldots, 4$, therefore we will only need to shift the vector $\vec{s}_2$ four times.

The cross-correlation between two vectors is defined as follows:

$$\text{corr}(\vec{x}, \vec{y})[k] = \sum_{i=-\infty}^{\infty} \vec{x}[i]\vec{y}[i-k]$$

To compute the cross-correlation $\text{corr}(\vec{s}_1, \vec{s}_2)$, we shift the vector $\vec{s}_2$ and compute the inner product of the shifted $\vec{s}_2$ and the vector $\vec{s}_1$.

$$
\begin{array}{c|cccccccc}
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
\vec{s}_2[n] & -4 & 8 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\langle\vec{s}_1, \vec{s}_2[n]\rangle & -16 & +16 & +0 & +0 & +0 & +0 & +0 & +0 & +0 = -32 \\
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
\vec{s}_2[n-1] & 0 & -4 & -8 & -4 & 0 & 0 & 0 & 0 & 0 \\
\langle\vec{s}_1, \vec{s}_2[n-1]\rangle & 0 & +8 & +8 & +0 & +0 & +0 & +0 & +0 & +0 = 8 \\
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
\vec{s}_2[n-2] & 0 & 0 & -4 & 8 & -4 & 0 & 0 & 0 & 0 \\
\langle\vec{s}_1, \vec{s}_2[n-2]\rangle & 0 & +0 & +0 & +0 & +8 & +0 & +0 & +0 & +0 = 8 \\
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
\vec{s}_2[n-3] & 0 & 0 & 0 & -4 & 8 & -4 & 0 & 0 & 0 \\
\langle\vec{s}_1, \vec{s}_2[n-3]\rangle & 0 & +0 & +0 & +0 & +16 & +0 & +0 & +0 & +0 = -16 \\
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
\vec{s}_2[n-4] & 0 & 0 & 0 & 0 & -4 & 8 & -4 & 0 & 0 \\
\langle\vec{s}_1, \vec{s}_2[n-4]\rangle & 0 & +0 & +0 & +0 & +8 & +0 & +0 & +0 & +0 = 8
\end{array}
$$
(b) Now the pattern in $\vec{s}_1$ is repeated three times:

![Sketch of signals](image)

Sketch the linear cross-correlation of signal 1 with signal 2, $\text{corr}(\vec{s}_1, \vec{s}_2)[n]$, for $n = 0, 1, \ldots, 4$.

**Answer:** Recall that $\text{corr}(\vec{x}, \vec{y})[k] = \sum_{i=-\infty}^{\infty} \vec{x}[i] \vec{y}[i-k]$

As we did in part a) to compute the cross-correlation $\text{corr}(\vec{s}_1, \vec{s}_2)$, we shift the vector $\vec{s}_2$ and compute the inner product of the shifted $\vec{s}_2$ and the vector $\vec{s}_1$. Since we are interested in $\text{corr}(\vec{s}_1, \vec{s}_2)[n]$, for $n = 0, 1, \ldots, 4$, here we have shown the two signals for $n = 0, 1, \ldots, 8$.

<table>
<thead>
<tr>
<th>$\vec{s}_1$</th>
<th>4</th>
<th>-2</th>
<th>0</th>
<th>0</th>
<th>-2</th>
<th>-2</th>
<th>0</th>
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<th>-2</th>
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</thead>
<tbody>
<tr>
<td>$\vec{s}_2[n]$</td>
<td>-4</td>
<td>8</td>
<td>-4</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\langle \vec{s}_2, \vec{s}_1[n]\rangle$</td>
<td>-16</td>
<td>+16</td>
<td>+0</td>
<td>+0</td>
<td>+0</td>
<td>+0</td>
<td>+0</td>
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</table>

$= -32$

<table>
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<tr>
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<th>-2</th>
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<th>0</th>
<th>-2</th>
<th>-2</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\vec{s}_2[n-1]$</td>
<td>0</td>
<td>-4</td>
<td>8</td>
<td>-4</td>
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<td>0</td>
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<tr>
<td>$\langle \vec{s}_2, \vec{s}_1[n-1]\rangle$</td>
<td>0</td>
<td>+8</td>
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$= 8$

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<td>$\langle \vec{s}_2, \vec{s}_1[n-3]\rangle$</td>
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<tbody>
<tr>
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<td>0</td>
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<td>-4</td>
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<tr>
<td>$\langle \vec{s}_2, \vec{s}_1[n-4]\rangle$</td>
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<td>+8</td>
<td>+32</td>
<td>+8</td>
<td>+0</td>
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</tbody>
</table>

$= 48$
Notice that when $\overline{s}_1$ is periodic we don’t simply get the result from part a) repeated.