1. Visualizing Span

We are given a point \( \vec{c} \) that we want to get to, but we can only move in two directions: \( \vec{a} \) and \( \vec{b} \). We know that to get to \( \vec{c} \), we can travel along \( \vec{a} \) for some amount \( \alpha \), then change direction, and travel along \( \vec{b} \) for some amount \( \beta \). We want to find these two scalars \( \alpha \) and \( \beta \), such that we reach point \( \vec{c} \). That is, \( \alpha \vec{a} + \beta \vec{b} = \vec{c} \).

![Diagram showing vectors \( \vec{a}, \vec{b}, \vec{c} \)]

(a) First, consider the case where \( \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \) and \( \vec{z} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \). Draw these vectors on a sheet of paper.

(b) We want to find the two scalars \( \alpha \) and \( \beta \), such that by moving \( \alpha \) along \( \vec{x} \) and \( \beta \) along \( \vec{y} \) so that we can reach \( \vec{z} \). Write a system of equations to find \( \alpha \) and \( \beta \) in matrix form.

(c) Solve for \( \alpha, \beta \).

2. Span basics

(a) What is span \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\} \)?

(b) Is \( \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} \) in span \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\} \)?

(c) What is a possible choice for \( \vec{v} \) that would make span \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3 \)?

(d) For what values of \( b_1, b_2, b_3 \) is the following system of linear equations consistent? (“Consistent” means there is at least one solution.)

\[
\begin{bmatrix}
1 & 2 \\
2 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\vec{x} \\
\vec{b}
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]
3. Proofs

**Definition:** A set of vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) is **linearly dependent** if there exists constants \( c_1, c_2, \ldots, c_n \) such that \( \sum_{i=1}^{n} c_i \vec{v}_i = \vec{0} \) and at least one \( c_i \) is non-zero.

This condition intuitively states that it is possible to express any vector from the set in terms of the others.

(a) Suppose for some non-zero vector \( \vec{x} \), \( A \vec{x} = \vec{0} \). Prove that the columns of \( A \) are linearly dependent.

(b) For \( A \in \mathbb{R}^{m \times n} \), suppose there exist two unique vectors \( \vec{x}_1 \) and \( \vec{x}_2 \) that both satisfy \( A \vec{x} = \vec{b} \), that is, \( A \vec{x}_1 = \vec{b} \) and \( A \vec{x}_2 = \vec{b} \). Prove that the columns of \( A \) are linearly dependent.

(c) Let \( A \in \mathbb{R}^{m \times n} \) be a matrix for which there exists a non-zero \( \vec{y} \in \mathbb{R}^n \) such that \( A \vec{y} = \vec{0} \). Let \( \vec{b} \in \mathbb{R}^m \) be some non-zero vector. Show that if there is one solution to the system of equations \( A \vec{x} = \vec{b} \), then there are infinitely many solutions.