1. Steady and Unsteady States

(a) You’re given the matrix $M$:

$$
M = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
0 & 1 & -2 \\
0 & 0 & 2
\end{bmatrix}
$$

Which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = M\vec{x}[k]$. ($\vec{x}$ could describe either people or water.) Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$

ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$

iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

(b) Define $\vec{x} = \alpha \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3$, a linear combination of the eigenvectors. For each of the cases in the table, determine if

$$
\lim_{n \to \infty} M^n \vec{x}
$$

converges. If it does, what does it converge to?
2. **Eigenvalues and Special Matrices – Visualization**

As seen earlier, an eigenvector $\vec{v}$ belonging to a square matrix $A$ is a nonzero vector that satisfies

$$A\vec{v} = \lambda \vec{v}$$

where $\lambda$ is a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?
(b) Does a diagonal matrix
\[
\begin{bmatrix}
  d_1 & 0 & 0 & \cdots & 0 \\
  0 & d_2 & 0 & \cdots & 0 \\
  0 & 0 & d_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & d_n
\end{bmatrix}
\]
in \( \mathbb{R}^n \) have any eigenvalues \( \lambda \in \mathbb{R} \)? What are the corresponding eigenvectors?

(c) Conceptually, does a rotation matrix in \( \mathbb{R}^2 \) by angle \( \theta \) have any eigenvalues \( \lambda \in \mathbb{R} \)? For which angles is this the case?
(d) Now let us mechanically compute the eigenvalues of the rotation matrix in $\mathbb{R}^2$. Does it agree with our findings above? As a refresher, the rotation matrix $R$ has the following form:

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

(e) Does the reflection matrix $T$ across the x-axis in $\mathbb{R}^{2\times2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
(f) If a matrix $M$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $M\vec{x} = \vec{b}$?

(g) (Practice) Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?