1. Steady and Unsteady States

(a) You’re given the matrix \( M \):

\[
M = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
0 & 1 & -2 \\
0 & 0 & 2
\end{bmatrix}
\]

Which generates the next state of a physical system from its previous state: \( \vec{x}[k+1] = M \vec{x}[k] \). Find the eigenspaces associated with the following eigenvalues:

i. span(\( \vec{v}_1 \)), associated with \( \lambda_1 = 1 \)
ii. span(\( \vec{v}_2 \)), associated with \( \lambda_2 = 2 \)
iii. span(\( \vec{v}_3 \)), associated with \( \lambda_3 = \frac{1}{2} \)

(b) Define \( \vec{x} = \alpha \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3 \), a linear combination of the eigenvectors. For each of the cases in the table, determine if

\[
\lim_{n \to \infty} M^n \vec{x}
\]

converges. If it does, what does it converge to?

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<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>Converges?</th>
<th>( \lim_{n \to \infty} M^n \vec{x} )</th>
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2. Steady State Reservoir Levels

We have 3 reservoirs: \( A, B \) and \( C \). The pumps system between the reservoirs is depicted in Figure 1.
(a) Write out the transition matrix $T$ representing the pumps system.

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = -\sqrt{2} - 1$, $\lambda_3 = \sqrt{2} - 1$ are the eigenvalues of $T$. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

(c) What does the magnitude of the other two eigenvalues $\lambda_2$ and $\lambda_3$ say about the steady state behavior of their associated eigenvectors?

(d) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129$, $B_0 = 109$, $C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?