EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 6A

1. **(Optional) The Romulan Ruse** While scanning parts of the galaxy for alien civilization, the starship USS Enterprise NC-1701D encounters a Romulan starship that is known for advanced cloaking devices.

   (a) *Concept: Matrix Transformations*

   The Romulan illusion technology causes a point (x_0, y_0) to transform or *map* to (u_0, v_0). Similarly, (x_1, y_1) is mapped to (u_1, v_1). Figure 1 and Table 1 show two points on a Romulan ship and the corresponding *mapped* points.

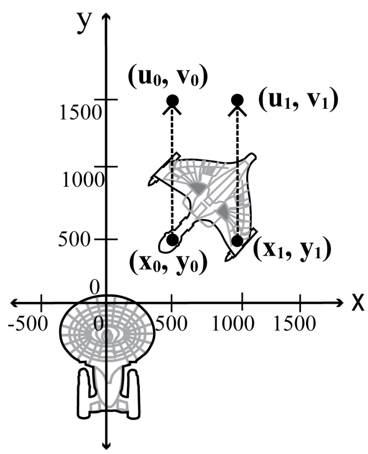

Figure 1: Figure for part (a)

Original Point	Mapped Point
$(x_0, y_0) = (500, 500)$	$(u_0, v_0) = (500, 1500)$

Original Point	Mapped Point
$(x_1, y_1) = (1000, 500)$	$(u_1, v_1) = (1000, 1500)$

Table 1: Original and Mapped Points

Find a transformation matrix $\mathbf{A}_0$ such that

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \text{ and } \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

(b) *Concept: Matrix Transformations*

In this scenario, every point on the Romulan ship (x_m, y_m) is mapped to (u_m, v_m), such that vector $\begin{bmatrix} x_m \\ y_m \end{bmatrix}$ is rotated counterclockwise by 30° and then scaled by 2 in the x- and y-directions. This transformation is shown in Figure 2.

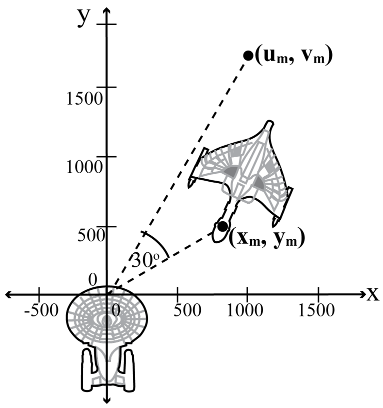

Figure 2: Figure for part (b)

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	∞

Table 2: Trigononometric Table

Find a transformation matrix R such that $\begin{bmatrix} u_m \\ v_m \end{bmatrix} = \mathbf{R} \begin{bmatrix} x_m \\ y_m \end{bmatrix}$**.**

The Romulan ship has launched a probe into space and the Enterprise is trying to destroy the probe by firing a photon torpedo along a straight line from point $(0,0)$ towards the probe.

(c) *Concept: Gaussian Elimination, Systems of Equations*

The Romulan generals found a clever way to hide the probe by transforming (mapping) its position with a *cloaking* (transformation) matrix $\mathbf{A}_p$:

$$\mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

They positioned the probe at (x_p, y_p) so that it maps to $(u_p, v_p) = (0,0)$, where $\begin{bmatrix} u_p \\ v_p \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_p \\ y_p \end{bmatrix}$. This scenario is shown in Figure 3. The initial position of the torpedo is $(0,0)$ and the torpedo cannot be fired on its initial position! Impressive trick indeed!

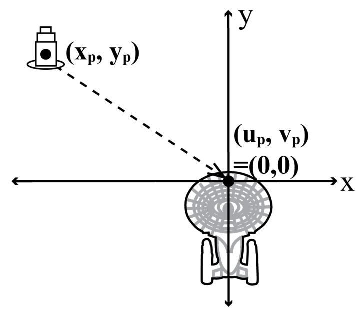

Figure 3: Figure for part (c)

Find the possible positions of the probe (x_p, y_p) **so that** $(u_p, v_p) = (0,0)$**.**

(d) *Concept: Eigenspaces/Eigenvectors/Eigenvalues*

It turns out the Romulan engineers were not as smart the Enterprise engineers. Their calculations did not work out and they positioned the probe at (x_q, y_q) such that the *cloaking* (transformation) matrix, $\mathbf{A}_p$, mapped it to (u_q, v_q), where

$$\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_q \\ y_q \end{bmatrix}, \text{ and } \mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

As a result, the torpedo while traveling along a straight line from $(0,0)$ to (u_q, v_q), hit the probe at (x_q, y_q) on the way!

The scenario is shown in Figure 4. For the torpedo to hit the probe, we must have $\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$, where λ is a real number.

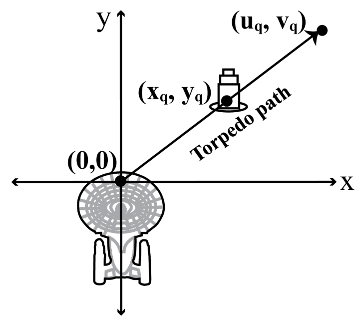

Figure 4: Figure for part (d)

Find the possible positions of the probe (x_q, y_q) so that $(u_q, v_q) = (\lambda x_q, \lambda y_q)$. Remember that the torpedo cannot be fired on its initial position $(0,0)$.

2. (Optional) Proof

Concept: Null Spaces, Invertibility

Consider a square matrix $\mathbf{A}$. Prove that if $\mathbf{A}$ has a non-trivial nullspace, i.e. if the nullspace of $\mathbf{A}$ contains more than just $\vec{0}$, then matrix $\mathbf{A}$ is not invertible.