EECS 16A Designing Information Devices and Systems I
Spring 2023 Exam Prep 6A

1. Mixing Paints (Fall 2020 Midterm 1 Question 4)

Over quarantine, you've really gotten into painting, but you are running low on paints.You would like to mix the paints you have to make different colors.

Notation: **Every** color can be represented as a length-3 vector, $\vec{c} = \begin{bmatrix} c_r \\ c_y \\ c_b \end{bmatrix}$, where c_r, c_y, and c_b represent the number of bottles of red, yellow, and blue paint, respectively, to make the color.

(a) You want to make the color brown, $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, using a linear combination of turquoise, magenta, and peach paints (the only ones you have left). We are given the following information about paint compositions:

- 1 bottle of turquoise is made by combining 0 bottles red, 0.4 bottles yellow, and 0.6 bottles blue.
- 1 bottle of magenta is made by combining 0.5 bottles red, 0 bottles yellow, and 0.5 bottles blue.
- 1 bottle of peach is made by combining 0.5 bottles red, 0.4 bottles yellow, and 0.1 bottles blue.

You would like to find the number of bottles of turquoise (x_t), magenta (x_m), and peach (x_p) to mix to make brown. Formulate this problem as a matrix-vector equation. You do not need to solve the equation.

(b) Your friend wants to make the color brown but has a different set of colors. She also sets up a system of equations to make brown and ends up with $\mathbf{A}\vec{x} = \vec{b}$ where

$$\mathbf{A} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Use Gaussian Elimination to solve for $\vec{x}$, which represents the mixture of paints your friend must combine to make brown. Show your work.

(c) You now want to make lots of colors. Instead of solving the system $\mathbf{A}\vec{x} = \vec{c}$ every time we want to make a new color $\vec{c}$, you want to find the inverse matrix $\mathbf{A}^{-1}$. For this subpart, use the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Use Gaussian Elimination to determine if $\mathbf{A}^{-1}$ exists. If $\mathbf{A}^{-1}$ exists, give its value.

(d) We would like to determine what colors we can make just given three other colors (purple, orange, gray). These are represented by a matrix $\mathbf{D} \in \mathbb{R}^{3\times 3}$, which is invertible, but is otherwise unknown to you. You cannot mix together negative amounts of paint.

Consider the set of all colors

$$\mathbb{S} = \left\{ \vec{c} \in \mathbb{R}^3 \mid \quad \mathbf{D}\vec{x} = \vec{c},\ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}.$$

that we can get from mixing together non-negative quantities of the three colors. **Is $\mathbb{S}$ a vector subspace of $\mathbb{R}^3$? Justify your answer.**

2. Dynamical Systems (Spring 2020 Midterm 1 Question 7)

Define matrices $Q, R \in \mathbb{R}^{2\times 2}$ according to

$$Q = \begin{bmatrix} 0 & 3/4 \\ 1 & 1/4 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues for the matrix Q.

(b) Consider a system with state vector $\vec{x}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by

$$\vec{x}[n] = Q\vec{x}[n-1].$$

Is there a non-zero vector $\vec{x}$ satisfying $\vec{x} = Q\vec{x}$? If yes, give one such vector.

(c) Draw the state-transition diagram for the system in part (b). Label your nodes "A" and "B".

(d) Now, consider a system with state vector $\vec{w}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by:

$$\vec{w}[n] = \begin{cases} Q\,\vec{w}[n-1] & \text{if } n \text{ is odd} \\ R\,\vec{w}[n-1] & \text{if } n \text{ is even.} \end{cases}$$

Write expressions for $\vec{w}[1]$, $\vec{w}[2]$, $\vec{w}[3]$ and $\vec{w}[4]$ in terms of $\vec{w}[0]$ and Q and R. Write each answer in the form of a matrix-vector product.

(e) Suppose we start the system of part (d) with state $\vec{w}[0] = \begin{bmatrix} 11/14 & 3/14 \end{bmatrix}^\top$. Find expressions for $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$, which are defined according to

$$\vec{w}_{\text{even}} = \lim_{k\to\infty} \vec{w}[2k], \qquad \vec{w}_{\text{odd}} = \lim_{k\to\infty} \vec{w}[2k+1].$$

In words, $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$ describe the long-term behavior of the system at even and odd time-instants, respectively. (Hint: you can avoid computation by thinking about the system at even time-instants in terms of a state-transition diagram.)

3. Proof (Fall 2019 Midterm 1 Question 9)

Consider a square matrix $\mathbf{A}$. Prove that if $\mathbf{A}$ has a non-trivial nullspace, i.e. if the nullspace of $\mathbf{A}$ contains more than just $\vec{0}$, then matrix $\mathbf{A}$ is not invertible.

Justify every step. Proofs that are not properly justified will not receive full credit. Simply invoking a theorem such as the "Invertible Matrix Theorem" will receive no credit.