Dis 11B EECS16A

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Reference: Inner products

For this course we will use a standard inner product definition from matrix-vector multiplication:

$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_v, \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n.$$

In general, any inner product $\langle \cdot, \cdot \rangle$ on a real vector space $\mathbb{V}$ is a bilinear function that satisfies the following three properties:

(a) **Symmetry:** $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

(b) **Linearity:** $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$ and $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$, where $c \in \mathbb{R}$ is a real number.

(c) **Non-negativity:** $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$.

Here $\vec{x}$, $\vec{y}$, and $\vec{z}$ can be any vectors in the vector space $\mathbb{V}$.
The norm (or length) of a vector $\vec{x} = [x_1, x_2, \ldots, x_n]^T$ is defined using the inner product as

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} \equiv \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

$\langle \vec{x}, \vec{y} \rangle$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ dim: #rows x #cols

$$\underbrace{\langle \vec{x}, \vec{y} \rangle}_{\text{Euclidean inner product}} = \vec{x}^T\vec{y} = \underbrace{\begin{bmatrix} x_1 & \cdots & x_2 \end{bmatrix}}_{1 \times n} \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{n \times 1} = 1 \times 1$$

$$= x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

1. Inner Product Properties

For this question we will verify our coordinate definition of the inner product

$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_v, \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n$$

indeed satisfies the key properties required for all inner products, but presently for the 2-dimensional case. Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ for the following parts:

(a) Show symmetry $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$: $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 = y_1x_1 + y_2x_2 = \langle \vec{y}, \vec{x} \rangle \quad \checkmark$$

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T\vec{y} = \vec{x} \cdot \vec{y}$

* also known as a dot product

(b) Show linearity $\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle$, where $c \in \mathbb{R}$ is a real number.

c, d scalars $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} cy_1 + dz_1 \\ cy_2 + dz_2 \end{bmatrix} \right\rangle = x_1(cy_1 + dz_1) + x_2(cy_2 + dz_2)$$
$$= c(x_1y_1 + x_2y_2) + d(x_1z_1 + x_2z_2)$$
$$= c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle$$

(c) Show non-negativity $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$:

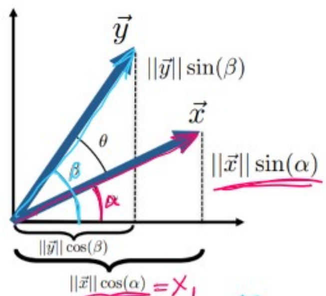
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \langle \vec{x}, \vec{x} \rangle = \underbrace{x_1^2 + x_2^2}_{\text{no choices of } x_1, x_2 \text{ to make this negative}} \geq 0$$

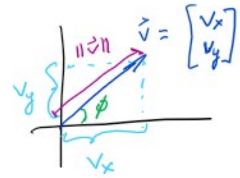
* only 0 if $x_1 = x_2 = 0$ or $\vec{x} = \vec{0}$

2. Geometric Interpretation of the Inner Product

In this problem we explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in $\mathbb{R}^2$.

(a) Derive a formula for the inner product of two vectors in terms of their magnitudes and the angle between them. The figure below may be helpful:





$||\vec{v}||$ length of the vector

$||\vec{v}|| = \sqrt{v_x^2 + v_y^2}$

$$\begin{cases} v_x = ||\vec{v}|| \cos\phi \\ v_y = ||\vec{v}|| \sin\phi \end{cases}$$

Sum and Difference Identities

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\sin(a-b) = \sin a \cos b - \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$

$\tan(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$

$\cos(\phi) = \cos(-\phi)$

$\vec{x} = \begin{bmatrix} ||\vec{x}|| \cos\alpha \\ ||\vec{x}|| \sin\alpha \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} ||\vec{y}|| \cos\beta \\ ||\vec{y}|| \sin\beta \end{bmatrix}$

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= (||\vec{x}|| \cos\alpha)(||\vec{y}|| \cos\beta) + (||\vec{x}|| \sin\alpha)(||\vec{y}|| \sin\beta) \\ &= ||\vec{x}|| \, ||\vec{y}|| \, (\cos\alpha \cos\beta + \sin\alpha \sin\beta) \\ &= ||\vec{x}|| \, ||\vec{y}|| \cos(\alpha - \beta) = ||\vec{x}|| \, ||\vec{y}|| \cos(\beta - \alpha) \end{aligned}$$

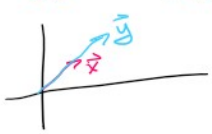
$$\boxed{\langle \vec{x}, \vec{y} \rangle = ||\vec{x}|| \, ||\vec{y}|| \cos(\theta)}$$

$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2$

(b) For each sub-part, identify any two (nonzero) vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ that satisfy the stated condition and compute their inner product.

i. Identify a pair of parallel vectors:

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

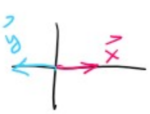


$\langle \vec{x}, \vec{y} \rangle = (1)(2) + (1)(2) = 4$

$\langle \vec{x}, \vec{y} \rangle = ||\vec{x}|| \, ||\vec{y}|| \cos(\underset{\theta = 0}{\theta}) = (\sqrt{1^2 + 1^2})(\sqrt{2^2 + 2^2}) \underbrace{\cos(0)}_{1} = (\sqrt{2})(\sqrt{8}) = \sqrt{16} = 4$

ii. Identify a pair of anti-parallel vectors:

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$



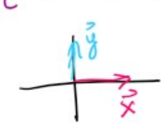
$\langle \vec{x}, \vec{y} \rangle = (1)(-1) + (0)(0) = -1$

$\langle \vec{x}, \vec{y} \rangle = ||\vec{x}|| \, ||\vec{y}|| \cos\theta = (\sqrt{1^2})(\sqrt{1^2})(-1) = -1$

$\theta = 180°$

iii. Identify a pair of perpendicular vectors:

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$\langle \vec{x}, \vec{y} \rangle = (1)(0) + (0)(1) = 0$

$\langle \vec{x}, \vec{y} \rangle = ||\vec{x}|| \, ||\vec{y}|| \cos\theta = (\sqrt{1^2})(\sqrt{1^2})(0) = 0$

$90°$

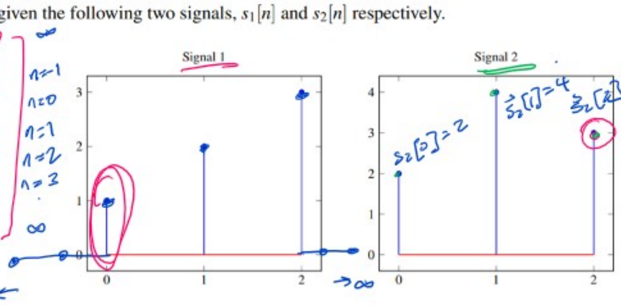
all perpendicular pairs of vectors will have an inner product of zero

orthogonal

3. Correlation

We are given the following two signals, $s_1[n]$ and $s_2[n]$ respectively.

$$\vec{s}_1 = \begin{array}{l} \\ n=-1 \\ n=0 \\ n=1 \\ n=2 \\ n=3 \\ \\ \end{array}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\begin{array}{l} \infty \\ \\ \\ \\ \\ \\ \infty \end{array} \qquad \vec{s}_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 2 \\ 4 \\ 3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\begin{array}{l} -\infty \\ n=-1 \\ n=0 \\ n=1 \\ n=2 \\ n=3 \\ \infty \end{array}$$



Find the cross correlations, $\text{corr}_{s_1}(s_2)$ and $\text{corr}_{s_2}(s_1)$ for signals $s_1[n]$ and $s_2[n]$. Recall

vector that gets shifted, shift, $i=n$

$$\text{corr}_x(y)[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k].$$

$k=-2$

shift $\vec{s}_2$

$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$

$\vec{s}_1[0]=1$

$\vec{s}_2[0+2]=3$

$k=-2$

$n=$	-2	-1	0	1	2	3	4	
$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n+2]$	2	4	3	0	0	0	0	
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	0 +	0 +	3 +	0 +	0 +	0 +	0	= 3

$k=-1$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n+1]$	0	2	4	3	0	0	0	
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	0 +	0 +	4 +	6 +	0 +	0 +	0	= 10

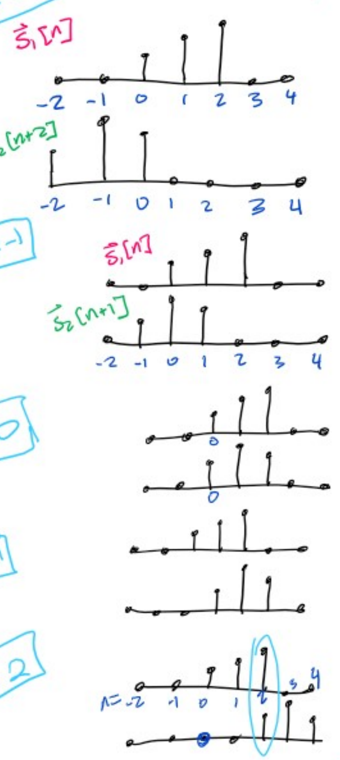
$k=0$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	0 +	0 +	2 +	8 +	9 +	0 +	0	= 19

$k=1$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n-1]$	0	0	0	2	4	3	0	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0 +	0 +	0 +	4 +	12 +	0 +	0	= 16

$n=$	-2	-1	0	1	2	3	4	
$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n-2]$	0	0	0	0	2	4	3	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0 +	0 +	0 +	0 +	6 +	0 +	0	= 6

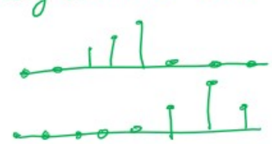


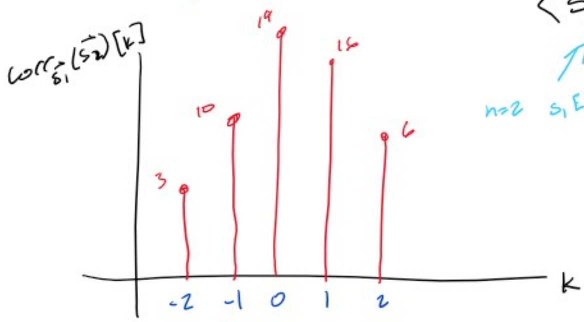
$\langle \vec{s}_1[n], \vec{s}_2[n-2] \rangle$

$n=2$ $s_1[2]$ $s_2[0]$

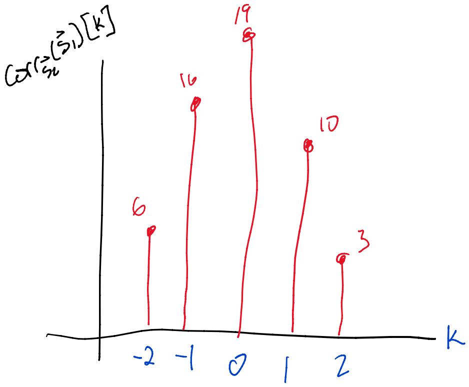
$$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k] = \sum_{i=-\infty}^{\infty} s_1[n]\, s_2[n-k]$$

why don't we check $k=3$





$$\left\langle \begin{array}{c} n=-2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}\begin{array}{c} -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \right\rangle = 6$$



shift $\vec{s}_1$

$\mathrm{corr}_{\vec{s}_2}(\vec{s}_1)[k]$

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n+2]$							
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$	+	+	+	+	+	+	=

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n+1]$							
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$	+	+	+	+	+	+	=

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n]$							
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	+	+	+	+	+	+	=

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n-1]$							
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	+	+	+	+	+	+	=

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n-2]$							
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	+	+	+	+	+	+	=