Dis 5A EECS16A

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1. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

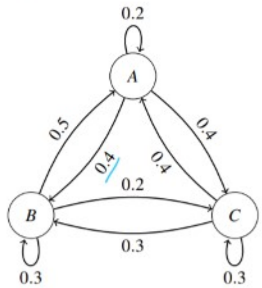


Figure 1: Reservoir pumps system.

(a) Write out the transition matrix **T** representing the pumps system.

$$x_A[n+1] = 0.2\,x_A[n] + 0.5\,x_B[n] + 0.4\,x_C[n]$$
$$x_B[n+1] = 0.4\,x_A[n] + 0.3\,x_B[n] + 0.3\,x_C[n]$$
$$x_C[n+1] = 0.4\,x_A[n] + 0.2\,x_B[n] + 0.3\,x_C[n]$$

$$\begin{bmatrix} x_A[n+1] \\ x_B[n+1] \\ x_C[n+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}}_{T} \begin{bmatrix} x_A[n] \\ x_B[n] \\ x_C[n] \end{bmatrix}$$

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of **T**. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

steady-state: not changing w/ time i.e. $\vec{x}[n+1] = \vec{x}[n] = \vec{x}_{ss}$

$$\vec{x}[n+1] = T\vec{x}[n]$$
$$\vec{x}_{ss} = T\vec{x}_{ss}$$
$$(1)\,\vec{x}_{ss} = T\vec{x}_{ss}$$

eigenvector $\vec{x} \neq \vec{0}$

$$T\vec{x} = \vec{x}$$
$$T\vec{x} - \vec{x} = \vec{0}$$
$$T\vec{x} - I\vec{x} = \vec{0}$$
$$(T-I)\vec{x} = \vec{0}$$
$$\underbrace{}_{A}\;\vec{x} = \vec{b}$$

sol case? inf

$$T - I = \begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix}$$

if unique $\vec{x} = \vec{0}$, $A\vec{x} = \vec{0}$

$$A\vec{x} = \vec{0}$$
$$A(2\vec{x}) = 2A\vec{x} = 2\cdot\vec{0} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -0.8 & 0.5 & 0.4 & 0 \\ 0.4 & -0.7 & 0.3 & 0 \\ 0.4 & 0.2 & -0.7 & 0 \end{array}\right] \xrightarrow[\substack{R_2 \leftarrow 10R_2 \\ R_3 \leftarrow 10R_3}]{R_1 \leftarrow 10R_1} \left[\begin{array}{ccc|c} -8 & 5 & 4 & 0 \\ 4 & -7 & 3 & 0 \\ 4 & 2 & -7 & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow 2R_3]{R_2 \leftarrow 2R_2} \left[\begin{array}{ccc|c} -8 & 5 & 4 & 0 \\ 8 & -14 & 6 & 0 \\ 8 & 4 & -14 & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 + R_1} \left[\begin{array}{ccc|c} -8 & 5 & 4 & 0 \\ 0 & -9 & 10 & 0 \\ 0 & 9 & -10 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_2 + R_3} \left[\begin{array}{ccc|c} -8 & 5 & 4 & 0 \\ 0 & -9 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{-43}{36} & 0 \\ 0 & 1 & \frac{-10}{9} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - \frac{43}{36}x_3 = 0$ $\quad x_1 = \frac{43}{36}a$

$x_2 - \frac{10}{9}x_3 = 0$ $\quad x_2 = \frac{10}{9}a$

$x_3 = $ anything $= a$ $\quad x_3 = a$

$$\vec{x} = a\underbrace{\begin{bmatrix} \frac{43}{36} \\ \frac{10}{9} \\ 1 \end{bmatrix}} = b\underbrace{\begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix}}$$

eigenspace for $\lambda = 1$: span$\left\{\begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix}\right\}$

$$T\begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix} = \begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix} \qquad T\begin{bmatrix} \frac{43}{36} \\ \frac{10}{9} \\ 1 \end{bmatrix} = \begin{bmatrix} 43/36 \\ 10/9 \\ 1 \end{bmatrix}$$

(c) What does the magnitude of the other two eigenvalues λ_2 and λ_3 say about the steady state behavior of their associated eigenvectors?

$\lambda_2 = -0.241 \qquad |\lambda_2| < 1$

$\lambda_3 = 0.0414 \qquad |\lambda_3| < 1$

3 distinct eigenvalues = 3 lin ind eigenvectors

$$\vec{x}[0] = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3 = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}\begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix}$$

$$\begin{aligned}\vec{x}[n] = T^n\vec{x}[0] &= T^n(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3) \\ &= \alpha T^n\vec{v}_1 + \beta T^n\vec{v}_2 + \gamma T^n\vec{v}_3 \\ &= \alpha\lambda_1^n\vec{v}_1 + \beta\lambda_2^n\vec{v}_2 + \gamma\lambda_3^n\vec{v}_3 \\ &= \alpha(1)^n\vec{v}_1 + \beta(-0.241)^n\vec{v}_2 + \gamma(0.0414)^n\vec{v}_3\end{aligned}$$

way in the future

$$\vec{x}_{ss} = \lim_{n\to\infty}\vec{x}[n] = \lim_{n\to\infty}\left(\alpha(1)^n\vec{v}_1 + \beta(-0.241)^n\vec{v}_2 + \gamma(0.0414)^n\vec{v}_3\right) = \alpha\vec{v}_1$$

converge for any α, β, γ

(d) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

$$\vec{x}[0] = \begin{bmatrix}129 \\ 109 \\ 0\end{bmatrix} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$$

+ 238 kL

don't know $\vec{v}_2, \vec{v}_3$

* notice cols sum to 1

→ conservative

→ no water is lost

$$\vec{x}_{ss} = \alpha\begin{bmatrix}43 \\ 40 \\ 36\end{bmatrix}$$

+ 238 kL

$43\alpha + 40\alpha + 36\alpha = 238$

$(43+40+36)\alpha = 238$

$119\alpha = 238$

$\alpha = 2$

$$\boxed{\vec{x}_{ss} = \begin{bmatrix}86 \\ 80 \\ 72\end{bmatrix}}$$

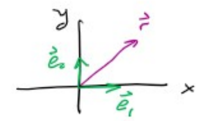
if not conservative have to find α, β, γ

2. REVIEW – Change of Basis

When we first encountered vectors, we defined $\vec{r} \in \mathbb{R}^n$ as *an ordered list of 'n' numbers*. Now we can see them in a new light; as coordinates corresponding to some set of 'directions' (which are also vectors!). The default basis is the set of *elementary column vectors* $\vec{e}_j$:

$$\vec{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \equiv r_1\vec{e}_1 + r_2\vec{e}_2 + \ldots + r_n\vec{e}_n = r_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + r_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \vec{r} = \mathbf{I}\,\vec{r}$$

$\text{span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2$



$$\vec{r} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{e}_1} + 2\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{e}_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = I\vec{r}$$

Now suppose we'd like to use a new set of basis vectors $\mathbb{V} := \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, which we can expand in a similar fashion!

$$\vec{r} \equiv r_1^{(v)}\vec{v}_1 + r_2^{(v)}\vec{v}_2 + \ldots + r_n^{(v)}\vec{v}_n = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} r_1^{(v)} \\ r_2^{(v)} \\ \vdots \\ r_n^{(v)} \end{bmatrix} = \mathbf{V}\,\vec{r}^{(v)}$$

$$\mathbb{V} := \text{span}\left\{\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{v}_2}\right\} = \mathbb{R}^2$$

$$\vec{r} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = r_1^{(v)}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + r_2^{(v)}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} r_1^{(v)} \\ r_2^{(v)} \end{bmatrix}}_{\vec{r}^{(v)}} = V\vec{r}^{(v)}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} r_1^{(v)} \\ r_2^{(v)} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

NOTE! Since we have required that $\mathbb{V}$ is a basis, we know (1) all vectors $\vec{v}_j$ are linearly independent from each other, and (2) it is a minimal set to span $\mathbb{R}^n$. From these properties we conclude that $\mathbf{V}$ is square and invertible! Thus we can identify the vector $\vec{r}^{(v)}$ expressed in the $\mathbb{V}$ basis from the original form of $\vec{r}$ using

$$\vec{r}^{(v)} = \mathbf{V}^{-1}\,\vec{r}\,.$$

$V\vec{r}^{(v)} = \vec{r}$

$\vec{r}^{(v)} = V^{-1}\vec{r}$

$\rightarrow \begin{bmatrix} r_1^{(v)} \\ r_2^{(v)} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

But what if (instead of the elementary basis) we had started in some other basis $\mathbb{U} := \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$, could we apply a transformation to $\vec{r}$ to map from the $\mathbb{U}$ basis to the $\mathbb{V}$ basis?
Yes we can!

$$\mathbb{U} := \text{span}\left\{\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$$

In fact, it follows quite naturally from our previous reasoning, just applied to a second basis.

$$\begin{aligned} \vec{r} = \quad r_1\,\vec{e}_1 + r_2\,\vec{e}_2 + \ldots + r_n\,\vec{e}_n &= \quad \mathbf{I}\,\vec{r} \\ r_1^{(v)}\vec{v}_1 + r_2^{(v)}\vec{v}_2 + \ldots + r_n^{(v)}\vec{v}_n &= \quad \mathbf{V}\,\vec{r}^{(v)} \\ r_1^{(u)}\vec{u}_1 + r_2^{(u)}\vec{u}_2 + \ldots + r_n^{(u)}\vec{u}_n &= \quad \mathbf{U}\,\vec{r}^{(u)} \end{aligned}$$

$\vec{r} = r_1\vec{e}_1 + r_2\vec{e}_2 = I\vec{r}$

$= r_1^{(v)}\vec{v}_1 + r_2^{(v)}\vec{v}_2 = V\vec{r}^{(v)}$

$= r_1^{(u)}\vec{u}_1 + r_2^{(u)}\vec{u}_2 = U\vec{r}^{(u)}$

From these equalities we can derive the transformation from $\vec{r}^{(v)}$ into $\vec{r}^{(u)}$ using $\mathbf{W} = \mathbf{V}^{-1}\,\mathbf{U}$ as shown below:

$$\vec{r}^{(v)} = \mathbf{V}^{-1}\mathbf{U}\;\vec{r}^{(u)}$$

$V\vec{r}^{(v)} = U\vec{r}^{(u)}$

$\vec{r}^{(v)} = \underbrace{V^{-1}U}_{W}\vec{r}^{(u)} \neq UV^{-1}\vec{r}^{(u)}$

3. Coordinate Change Examples

For the following sub-parts, two sets of basis vectors are specified $\mathbb{U} := \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and $\mathbb{V} := \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Provided the vector $\vec{r}$ expressed in the $\mathbb{U}$ basis $\vec{r}^{(u)}$, identify the matrix describing the transformation $\vec{r}^{(u)} \rightarrow \vec{r}^{(v)}$:

$\vec{r}^{(v)} = T\vec{r}^{(u)}$

$V\vec{r}^{(v)} = U\vec{r}^{(u)}$

$\vec{r}^{(v)} = \underbrace{V^{-1}U}_{T}\vec{r}^{(u)}$

(a) Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{?}{=} I \qquad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{r}^{(v)} = \mathbf{T}\vec{r}^{(u)}$ where $\vec{r}^{(u)}$ contains the coordinates of a vector in a basis of the columns of $\mathbf{U}$ and $\vec{r}^{(v)}$ is the coordinates of the same vector in the basis of the columns of $\mathbf{V}$.

Let $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{r}^{(v)}$. Repeat this for $\vec{r}^{(u)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now let $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What is $\vec{r}^{(v)}$?

$$V^{-1}: \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$T = V^{-1}U = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

for $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{r}^{(v)} = T\vec{r}^{(u)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$\vec{r}^{(u)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad \vec{r}^{(v)} = T\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{r}^{(v)} = T\vec{r}^{(u)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

(b) Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{r}^{(v)} = \mathbf{T}\vec{r}^{(u)}$.

Let $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and compute $\vec{r}^{(v)}$. Repeat this for $\vec{r}^{(u)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$V\vec{r}^{(v)} = U\vec{r}^{(u)}$$

$$\vec{r}^{(v)} = \underbrace{V^{-1}U}_{T}\vec{r}^{(u)}$$

$$* \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$V^{-1} = \frac{1}{(1)(1)-(1)(0)} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$T = V^{-1}U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}$$

for $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{r}^{(v)} = \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\vec{r}^{(u)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{r}^{(v)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(c) Let us return to thinking in general about two sets of basis vectors for $\mathbb{R}^n$, $\mathbb{U} := \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and $\mathbb{V} := \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Conversely, what is the coordinate transformation from $\vec{r}^{(v)}$ to $\vec{r}^{(u)}$? Find $\mathbf{Q}$ in terms of the two basis sets such that $\vec{r}^{(u)} = \mathbf{Q}\vec{r}^{(v)}$:

$$* \quad U^{-1}V\vec{r}^{(v)} = U^{-1}U\vec{r}^{(u)}$$

$$U^{-1}U\vec{r}^{(v)} = I\vec{r}^{(u)} = \vec{r}^{(u)}$$

$$V\vec{r}^{(v)} = U\vec{r}^{(u)}$$

$$\vec{r}^{(v)} = V^{-1}U\vec{r}^{(u)}$$

$$U^{-1}V\vec{r}^{(v)} = \vec{r}^{(u)}$$

(d) **(PRACTICE)** Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{r}^{(v)} = \mathbf{T}\vec{r}^{(u)}$. Let $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{r}^{(v)}$. Repeat this for $\vec{r}^{(u)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{r}^{(u)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{r}^{(v)}$?