1. Capacitance Equivalence

For the structures shown below, assume that the plates have a depth $d$ into the page and a width $W$ and are always a distance $d$ apart. The dielectric between the plates has absolute permittivity $\varepsilon$. For the following calculations, assume the capacitance is purely parallel plate, i.e., ignore fringing field effects.

(a) What is the capacitance of the structure shown below?

\[ C_a = \varepsilon \frac{A}{d} = \varepsilon \frac{W \cdot d}{d} = \varepsilon \frac{W}{d} \]

(b) Suppose that we take two such structures and put them next to each other as shown below. What is the capacitance of this new structure?

\[ C_b = \varepsilon \frac{A}{d} = \varepsilon \frac{W \cdot 2d}{d} = 2 \varepsilon \frac{W}{d} \]

(c) Now suppose that rather than connecting the together as shown above, we connect them with an ideal wire as shown below. What is the capacitance of this structure?

\[ C_c = C_a + C_a = 2 \varepsilon \frac{W}{d} = C_b \]

(d) Suppose that we now take two capacitors and connect them as shown below. What is the capacitance of the structure?

\[ C_d = \left( \frac{1}{C_a} + \frac{1}{C_a} \right)^{-1} = \frac{C_a}{C_a + C_a} = \frac{C_a}{2C_a} = \frac{1}{2} \varepsilon \frac{W}{d} \]

(e) What is the capacitance of the structure shown below?

\[ C_e = \varepsilon \frac{W \cdot d}{2d} = \varepsilon \frac{W}{2} \]
2. Voltages Across Capacitors

For the circuits given below, determine the voltage across each capacitor and calculate the charge and energy stored on each capacitor (assumed all capacitors start uncharged) and then we’ve let the system reach steady state. We are also given $C_1 = 1 \mu F$, $C_2 = 2 \mu F$, and $V_i = 1 V$.

Recall charge has units of Coulombs (C), and capacitance is measured in Farads (F) = \text{C/}	ext{V}.

It may also help to note metric prefixes: $\mu = 10^{-6}$.

(a)

\[ E = \frac{1}{2} CV^2 \]

\[ Q = CV \]

\[ Q = C_1 V_c = \left(\frac{1}{4} \mu F \right) \left(\frac{1}{4} V\right) = \frac{1}{4} \mu C \]

\[ E = \frac{1}{2} C_1 V_c^2 = \frac{1}{2} \left(\frac{1}{4} \mu F \right) \left(\frac{1}{4} V\right)^2 = \frac{1}{4} \mu J \]

\[ V_{c1} = V_1 - 0 = V_5 \]

\[ Q = C_1 V_{c1} = \left(\frac{1}{4} \mu F \right) \left(\frac{1}{4} V\right) = \frac{1}{4} \mu C \]

\[ E = \frac{1}{2} C_1 V_{c1}^2 = \frac{1}{2} \left(\frac{1}{4} \mu F \right) \left(\frac{1}{4} V\right)^2 = \frac{1}{4} \mu J \]

(b)

Helpful diagrams for considering the charges capacitors linked in series:

(without any initial charges)

Left: Our series capacitors may be modeled as one equivalent capacitor $C_{eq}$, which after some time is charged up by $V_5$, and then $V_1$ to have $+q$ on the top plate and $-q$ on the bottom plate.

Middle: We return to the 2 capacitor picture, but carry this insight of equivalent charge with us. Now the charge $+q$ is on the top plate of capacitor $C_1$, and $-q$ is on the bottom plate of capacitor $C_2$.

Right: Since capacitor plates have opposite & equal charges, we obtain this final right diagram.

As another conceptual check, we notice that the node between $C_1$ and $C_2$ is isolated from any other connections and should always remain charge neutral. From the diagram right we see this is maintained since $(+q) + (-q) = 0$.

\[ V_{c_{eq}} = V_5 \]

\[ Q = C_{eq} V_{c_{eq}} = \frac{C_1 C_2}{C_1 + C_2} V_5 = \left(\frac{3}{4} \mu F \right) \left(\frac{1}{4} V\right) = \frac{3}{4} \mu C \]

\[ E_{c_{eq}} = \frac{1}{2} C_{eq} V_{c_{eq}}^2 = \frac{1}{2} \left(\frac{3}{4} \mu F \right) \left(\frac{1}{4} V\right)^2 \]

\[ V_2 = C_2 V_{c_2} = \frac{\frac{3}{4} \mu C}{C_2} \]

\[ E_2 = \frac{1}{2} C_2 V_{c_2}^2 = \frac{1}{2} \left(\frac{3}{4} \mu F \right) \left(\frac{1}{4} V\right)^2 \]
3. Capacitors and Charge Conservation:

(a) Consider the circuit below with $C_1 = C_2 = 1\, \mu F$ and an open switch. Suppose that $C_1$ is initially charged to $+1\, \text{V}$ and that $C_2$ is charged to $-2\, \text{V}$. How much charge is on $C_1$ and $C_2$?

\[ Q_{1}^{in} = C_1 V_1 = 1\, \mu F \times 1\, \text{V} = 1\, \mu C \]
\[ Q_{2}^{in} = C_2 V_2 = 1\, \mu F \times (-2\, \text{V}) = -2\, \mu C \]

(b) Now the switch is closed (i.e., the capacitors are connected together.) What are the voltages across and the charges on $C_1$ and $C_2$?

\[ \text{charge is conserved on floating node}!! \]
\[ Q_{open} = Q_{1}^{in} + Q_{2}^{in} = 1\, \mu C + (-2\, \mu C) = -1\, \mu C \]
\[ 3\mu C = (C_1 + C_2)V \]
\[ V = \frac{3\mu C}{C_1 + C_2} = \frac{3\mu C}{2\, \mu F} = 1.5\, \text{V} \]
\[ Q_{1}^{in} = C_1 V = (1\, \mu F)(1.5\, \text{V}) = 1.5\, \mu C \]
\[ Q_{2}^{in} = C_2 V = (1\, \mu F)(-1.5\, \text{V}) = -1.5\, \mu C \]