1. **Visualizing Span**: Finding \( \alpha, \beta \in \mathbb{R} \) so that \( \alpha \mathbf{a} + \beta \mathbf{b} = \mathbf{c} \)

2. \( \text{Draw: } \mathbf{x} = [1], \mathbf{y} = [2], \text{ and } \mathbf{z} = [-2] \)
   
   \[
   \mathbf{c} \mathbf{x} - 4 \mathbf{y} = \mathbf{z}
   \]

3. \( \textbf{b/c} \) Solve for \( \alpha, \beta \in \mathbb{R} \) so that \( \alpha \mathbf{x} + \beta \mathbf{y} = \mathbf{z} \)

   \[
   \begin{align*}
   \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
   \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 2 \beta \\ \beta \end{bmatrix} &= \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
   \begin{bmatrix} \alpha + 2\beta \\ \alpha + \beta \end{bmatrix} &= \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
   \alpha + 2\beta &= -2 \\
   \alpha + \beta &= 2
   \end{align*}
   \]

   Matrix Form:
   \[
   \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}
   \]

   \( R_2 \rightarrow R_2 - R_1 \)

   \[
   \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \end{bmatrix}
   \]

   \( R_2 \rightarrow -R_2 \)

   \( \beta = -4 \)

   \( \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \end{bmatrix} \)

   \[
   1 \alpha + 2 \left(-\frac{1}{4}\right) = -2 \\
   \alpha = -2 + 8 \\
   \alpha = 6
   \]
2. The Cave (Nara & Cody): Find the light from each cave!

\[ \chi_1, \chi_2, \chi_3, \chi_4 \] 

\begin{align*}
\chi_1 + \chi_3 &= m_1 \\
\chi_1 + \chi_2 &= m_2 \\
\chi_2 + \chi_4 &= m_3 \\
\chi_3 + \chi_4 &= m_4
\end{align*}

\[ K \] 

\[ \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\chi_1 \\
\chi_2 \\
\chi_3 \\
\chi_4
\end{bmatrix} = \begin{bmatrix}
m_1 \\
m_2 \\
m_3 \\
m_4
\end{bmatrix} \]

a) \[ \vec{x} \] labels the cave lighting. Write out a matrix \( K \), so that \( K \vec{x} \) performs the masking process.

'Casual' Method (b):

\[ \chi_1 + \chi_2 + \chi_3 + \chi_4 = m_1 + m_3 \]

\[ \chi_1 + \chi_2 + \chi_3 + \chi_4 = m_2 + m_4 \]
b) Can we get a unique solution?

- No solution, if \( m_1 - m_2 + m_3 - m_4 \neq 0 \)
- Otherwise there are infinite solutions

Formal Method:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & | & m_1 \\
1 & 1 & 0 & 0 & | & m_2 \\
0 & 1 & 0 & 1 & | & m_3 \\
0 & 0 & 1 & 1 & | & m_4 \\
\end{bmatrix}
\]

(Augmented Form)

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & | & m_1 \\
0 & 1 & -1 & 0 & | & m_2 - m_1 \\
0 & 0 & 1 & 1 & | & m_3 - m_2 + m_1 \\
0 & 0 & 0 & 1 & | & m_4 \\
\end{bmatrix}
\]

\[
R_2 \rightarrow R_2 + R_1
\]

\[
R_4 \rightarrow R_4 - R_3
\]

Final Form

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & | & m_1 \\
0 & 1 & -1 & 0 & | & m_2 - m_1 \\
0 & 0 & 1 & 1 & | & m_3 - m_2 + m_1 \\
0 & 0 & 0 & 0 & | & m_4 - m_3 + m_2 - m_1 \\
\end{bmatrix}
\]

\[O = m_1 - m_2 + m_3 - m_4\]

No unique solution!
Suppose Nara makes a 5\textsuperscript{th} measurement. Now can we determine the light from each cave?

\[
\begin{bmatrix}
\frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4 &= m_5
\end{bmatrix}
\]

\[
\begin{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\frac{1}{2} & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
m_1 \\
m_2 \\
m_3 \\
m_4 \\
m_5
\end{bmatrix}
\]

measurement 5

\[
\begin{bmatrix}
0.5 & 1.0 \\
1.0 & 0.5
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = 
\begin{bmatrix}
m_1 \\
m_2 \\
m_3 \\
m_4 \\
m_5
\end{bmatrix}
\]

\[
x_4 = -m_5 + m_1 - \frac{1}{2}m_2 + \frac{3}{2}m_4
\]

\[
x_3 + (\quad) = m_3 - m_2 + m_1
\]
3. Gaussian Elimination Practice

a) \[
\begin{bmatrix}
3 & -1 & 2 & 1 \\
0 & 0 & 2 & 1 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\
0 & 0 & 1 & \frac{1}{2} \\
\end{bmatrix}
\]

b) True/False: There is never a unique solution if the number of equations don't match.

False! Our problem 2c above is a counter example!

Also, think of this...

It's clear that \( x = 1 \), \( y = 2 \) but there are 3 equations!!

2) \[
\begin{bmatrix}
2 & 0 & 4 \\
0 & 1 & 2 \\
1 & 2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix}
6 \\
-3 \\
3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 3 \\
0 & 1 & 2 & -3 \\
1 & 2 & 0 & 3 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 0 & 2 & 1 & 3 \\
0 & 1 & 2 & -3 \\
0 & 2 & -2 & 0 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 0 & 2 & 1 & 3 \\
0 & 1 & 2 & -3 \\
0 & 0 & -6 & 6 \\
\end{bmatrix}
\]

Unique Solution
\[ \begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 8 \\ 1 & 3 & 5 \end{bmatrix} \left[ \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} \right] = \left[ \begin{array}{c} 2 \\ 0 \\ 3 \end{array} \right] \]

\[ \begin{bmatrix} 1 & 4 & 2 \\ 0 & -2 & 6 \\ 1 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -3 \\ 1 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix} \]

No Solution!!

\[ \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \left[ \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} \right] = \left[ \begin{array}{c} 7 \\ 3 \\ 1 \end{array} \right] \]

\[ \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 1 \\ 0 & -2 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

(infinite) Solutions!