1. Column Spaces & Null Spaces

For some matrix $A^{m \times n}$,

- Column space = span of column vectors.
- Null space = span of vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{0}$.

Given the following matrices, get:

i. Column space
ii. Null space
iii. Row-reduced column space
iv. Does the column space form a basis?

In words, "col(A)" is the space of all $\vec{y}$ you can reach using $\vec{x}$ and multiplying $\vec{y} = A\vec{x}$.

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

i. $\text{Col}(A) = \text{span} \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$

ii. $\text{Null}(A) = \text{span} \{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$

iii. Yes

iv. No

Long-form:

$\exists \alpha \in \mathbb{R}$

$\{\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} | \alpha \in \mathbb{R}\}$

"Number of needed free parameters = dimension of null(A)."

"One free parameter needed."

$A\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 0(\alpha) = 0 \rightarrow x_1 = 0$

$\vec{x} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$

"Number of needed free parameters = dimension of null(A)."
\[ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

i. \( \text{col}(B) = \text{span}\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \} \)

ii. \( \text{null}(B) = \text{span}\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \)

iii. \( \text{col}(B_{RR}) = \text{span}\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \} \times \)

iv. \( x \)

\[ R_2 \rightarrow R_2 + (1)R_1 \]
\[ B\hat{x} = \hat{0} \]

\[ 0(\alpha) + 1 \cdot x_2 = 0 \rightarrow x_2 = 0 \]
\[ \hat{x} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \]

Visually, we see \( B \) maps any vector to the \( \text{col}(A) \) space \( \equiv \{ x = y \} \).
Thus any \( \hat{x} \) with \( y = x_2 = 0 \) will map to the origin.

\[ C = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \]

i. \( \text{col}(C) = \text{span}\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \} \)

ii. \( \text{null}(C) = \text{span}\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \)

iii. \( \text{col}(C_{RR}) = \text{span}\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \} \checkmark \)

iv. Yes, \( \text{col}(C) \) is basis of \( \mathbb{R}^2 \)

\[ R_2 \rightarrow R_2 + R_1 \]
\[ \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \]

\[ R_2 \rightarrow R_2 / 3 \]
\[ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]

\[ x_2 = 0 \]
\[ x_1 + 2(0) = 0 \rightarrow x_1 = 0 \]
\[ \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \]
\[ \mathbf{D} = \begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix} \]

i. \( \text{col(D)} = \text{span} \left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \end{bmatrix} \right\} \)

ii. \( \text{null(D)} = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\} \)

iii. \( \text{col(D)} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \)

iv. No \( \text{col(D)} \) is NOT a basis for \( \mathbb{R}^2 \)

\[ \mathbf{E} = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} \]

i. \( \text{col(E)} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ -3 \end{bmatrix} \right\} \)

ii. \( \text{null(E)} = \text{span} \left\{ \begin{bmatrix} -1/2 \\ -5/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \right\} \)

iii. \( \checkmark \)

iv. No, because \( \text{col(E)} \) has linearly-dependent vectors.

\[ \mathbf{Dx} = \mathbf{0} \]

\[ \begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix} \rightarrow \frac{R_1 \rightarrow R_1/2}{R_2 \rightarrow R_2 - 3R_1} \]

\[ x_2 = \alpha \]

\[ x_1 + 2(2)(\alpha) = 0 \]

\[ x_1 = 2\alpha \]

\[ \mathbf{x} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} \]

\[ \mathbf{Ex} = \mathbf{0} \]

\[ \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 5/2 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 10 & 1/2 & -7/2 \\ 0 & 1 & 5/2 & 1/2 \end{bmatrix} \]

\[ \begin{bmatrix} x_3 = \alpha \\ x_4 = \beta \end{bmatrix} \]

\[ x_1 + \frac{1}{2} \alpha - \frac{7}{2} \beta = 0 \]

\[ x_2 + \frac{5}{2} \alpha + \frac{1}{2} \beta = 0 \]

\[ \mathbf{x} = \begin{bmatrix} \frac{-3\alpha + 2\beta}{2} \\ \frac{-5\alpha - \frac{1}{2}\beta}{2} \\ \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-3\alpha}{2} \\ -\frac{5\alpha}{2} \\ 0 \end{bmatrix} \]

\[ = \alpha \begin{bmatrix} -\frac{3}{2} \\ -\frac{5}{2} \\ 1 \end{bmatrix} + \beta \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \]
Identifying a basis:
For each set of vectors...

i. Do they describe a basis?
ii. Is the basis for \( \mathbb{R}^3 \) or different?

\[ a \]
\[ V_1 = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \]

i. No, since this doesn't span \( \mathbb{R}^3 \)
ii. X

\[ b \]
\[ V_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

\[ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \]

i. Yes! \( V_2 \) is a basis of \( \mathbb{R}^3 \)
ii. Check

Since a) Spans \( \mathbb{R}^3 \)
b) Are linearly independent

Since the row-reduced form of the matrix (formed by the span) shows it has an inverse, the columns must be linearly independent!
\[ V_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

i. No! Not linearly independent.

ii. Not a basis at all.

- By using the machinery above, our row-reduced form here would render a zero row, showing that the span contains linearly dependent vectors.

- Yet, by inspection\[ \begin{bmatrix} u_1 - u_2 - u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \]

proving they're linearly dependent.

- Still \( \text{span}\{\bar{u}_1, \bar{u}_2\} \) are all linearly independent.

- \( \text{span}\{\bar{u}_2, \bar{u}_3\} \)

- \( \text{span}\{\bar{u}_1, \bar{u}_3\} \)

Visually, \( V_3 \) is this plane:

(Plane is coming towards us, sorry for the poor visual :/)

\[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]