EECS 16A Designing Information Devices and Systems I
Spring 2021 Homework 5

This homework is due Friday, February 26, 2021, at 23:59.
Self-grades are due Tuesday, March 2, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.
- **We strongly recommended that you submit your self-grades PRIOR to taking Midterm 1 on March 1, 2021, since looking at the solutions earlier will help you to study for the midterm.**

Submit the file to the appropriate assignment on Gradescope.

1. Reading Assignment

For this homework, please read Note 8 through 9. These notes will give you an overview of matrix subspaces and eigenvalues/eigenvectors. You are always welcome and encouraged to read beyond this as well.

2. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(d) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

3. Can You Hear the Shape of a Drum?

This problem is inspired by a popular problem posed by Mark Kac in his article "Can you hear the shape of a drum?"[1] Kac's question was about different shapes of drums. Here's what he wanted to know: if the shape of a drum defines the sound that's made when we strike it, can we listen to the drum and automatically infer its shape? Deep down, this is really a question about eigenvalues and eigenvectors of a matrix. The vibrational dynamics of a particularly shaped drum membrane can be captured by a system of linear equations represented by a matrix. The eigenvalues and eigenvectors of this matrix reveal interesting properties about the drum that will help us answer the question: can we hear its shape?

Before we answer this question, we will first consider a simpler problem of modeling the vibration of a one dimensional violin string.

We'll use a model of vibration in one dimension given by the equation,

$$\frac{d^2u(x)}{dx^2} + \lambda u(x) = 0 \tag{1}$$

Here, u is the amount of vertical displacement of the string at a particular location x, and λ is an unknown parameter (which will turn out to be an eigenvalue, as you will see). We can make the approximation:

$$\frac{d^2u}{dx^2} \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \tag{2}$$

where h is some small constant. This approximation just follows from the limit based definition of the derivative, and allows us to discretize a continuous problem. You can think of h as the distance between two points in the discretized model of the string.

(a) First, look at the diagram below that shows a 1D violin string. To analyze the problem, we will consider the vibration of the string at 3 points on the string (in orange), while the 2 end points remain fixed (in purple). Assume that the length of the string is 1 meter (even though that's kind of long for a violin...). Furthermore, we will assume that all 5 points are equally spaced ($\frac{1}{4}$ meters apart). There are therefore 3 unknowns: $u[1]$, $u[2]$, and $u[3]$ (note: 1, 2, 3 stand for the labels given to the points and not their respective distances from point 0). Use equations 1 and 2 with $h = \frac{1}{4}$ to derive a matrix vector equation that describes the vibration of the string at the 3 points.

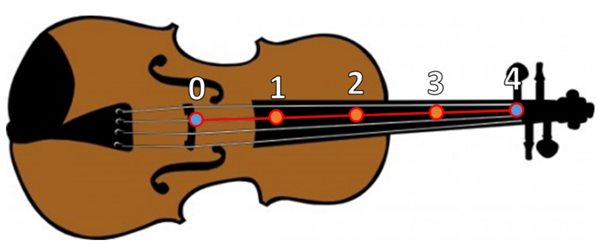

Figure 1: A 5-point model of a violin string.

(b) The 3 eigenvalues of the matrix in part a) happen to be

$$\lambda_1 = \frac{\sqrt{2}-2}{0.25^2} = -9.37...$$

[1] Marc Kac, Can one hear the shape of a drum?, Amer. Math. Monthly 73 (1966), 1-23.

$$\lambda_2 = \frac{-2}{0.25^2} = -32$$

$$\lambda_3 = \frac{-\sqrt{2}-2}{0.25^2} = -54.6...$$

For the vibrating string, find the 3 corresponding eigenvectors. What do these vectors look like?

(c) What do you think the eigenvalues mean for our vibrating string? (Hint: what does a larger eigenvalue seem to indicate about the corresponding eigenvector?)

In two dimensions, we can model vibration with the following equation instead:

$$\nabla^2 u(x,y) + \lambda u(x,y) = 0$$

The "∇^2" is an operator called the "Laplacian," and just stands for taking the 2nd x-partial-derivative and adding it to the 2nd y-partial-derivative. We can similarly approximate the laplacian operator with the following discretized difference equation:

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \frac{u(x+h,y)+u(x,y+h)-4u(x,y)+u(x,y-h)+u(x-h,y)}{h^2}$$

Using what you know from part (a) of this problem, we will write down the 5-point finite difference equation for a 5×5 square drum in the form of a matrix problem so that it has the same form as

$$-\lambda \vec{u} = \mathbf{A}\vec{u}$$

In this formulation, as in the 1D formulation, each row of $\mathbf{A}$ will correspond to the equation of motion for one point on the model. In our 5×5 grid, we will be modeling the motion of the inner 3×3 grid, since we will assume the membrane is fixed on the outer border. Since there are 9 points that we are modeling, this corresponds to 9 equations and 9 unknowns, so $\mathbf{A}$ should be 9×9.

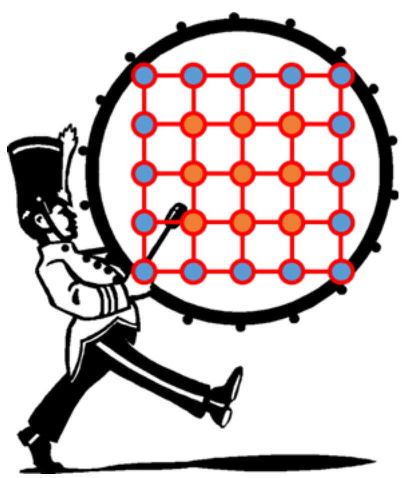

Figure 2: A 25-point model of a drum membrane.

(d) Based on our intuition from the 1D problem, what do the eigenvalues and eigenvectors correspond to in the 2D problem?

(e) Write down the 9×9 matrix, $\mathbf{A}$, for the drum in Figure 2. It should have some symmetry, but be careful with the diagonals.

(f) In the IPython Notebook, implement a function to solve the finite difference problem for a square drum of any side-length (though keep the side-length short at first, so that you don't run into memory problems!). What are the eigenvalues of the 5×5 drum?

(g) Using some of the built-in functionality in the notebook, you can construct a drum with any polygonal shape. There are two shapes already implemented, with the shapes shown below. The code already included will construct the $\mathbf{A}$ matrix given a polygon and a grid. Find the first 10 vibrational modes of each drum, and the associated eigenvalues (this is analogous to finding the first 10 eigenvectors of each $\mathbf{A}$ matrix, and the associated eigenvalues). Plot the 0th, 4th, and 8th modes using a contour plot.

(h) These two drums are different shapes. Do they sound the same? Why or why not? Can you hear the shape of a drum?

4. The Dynamics of Romeo and Juliet's Love Affair

Learning Goal: *Eigenvalues and eigenvectors of state transition matrices tend to reveal useful information about the dynamical systems they model. This problem serves as an example of extracting useful information through analysis of the eigenvalues of the state transition matrix of a dynamical system.*

In this problem, we will study a discrete-time model of the dynamics of Romeo and Juliet's love affair—adapted from Steven H. Strogatz's original paper, *Love Affairs and Differential Equations*, Mathematics Magazine, 61(1), p.35, 1988, which describes a continuous-time model.

Let $R[n]$ denote Romeo's feelings about Juliet on day n, and let $J[n]$ denote Juliet's feelings about Romeo on day n, where $R[n]$ and $J[n]$ are **scalars**. The **sign** of $R[n]$ (or $J[n]$) indicates like or dislike. For example, if $R[n] > 0$, it means Romeo likes Juliet. On the other hand, $R[n] < 0$ indicates that Romeo dislikes Juliet. $R[n] = 0$ indicates that Romeo has a neutral stance towards Juliet.

The **magnitude** (i.e. absolute value) of $R[n]$ (or $J[n]$) represents the intensity of that feeling. For example, a larger magnitude of $R[n]$ means that Romeo has a stronger emotion towards Juliet (strong love if $R[n] > 0$ or strong hatred if $R[n] < 0$). Similar interpretations hold for $J[n]$.

We model the dynamics of Romeo and Juliet's relationship using the following linear system:

$$R[n+1] = a\,R[n] + b\,J[n], \quad n = 0, 1, 2, \ldots$$

and

$$J[n+1] = c\,R[n] + d\,J[n], \quad n = 0, 1, 2, \ldots,$$

which we can rewrite as

$$\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n],$$

where $\vec{s}[n] = \begin{bmatrix} R[n] \\ J[n] \end{bmatrix}$ denotes the state vector and $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denotes the state transition matrix for our dynamic system model.

The selection of the parameters a, b, c, d results in different dynamic scenarios. The fate of Romeo and Juliet's relationship depends on these model parameters (i.e. a, b, c, d) in the state transition matrix and the initial state ($\vec{s}[0]$). In this problem, we'll explore some of these possibilities.

(a) Consider the case where $a+b=c+d$ in the state-transition matrix

$$\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that

$$\vec{v}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_1.
Show that

$$\vec{v}_2=\begin{bmatrix} b \\ -c \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_2.
Now, express the first and second eigenvalues and their eigenspaces in terms of the parameters $a, b, c,$ and d.
Hint: *Consider $\mathbf{A}\vec{v}_1$. Is it equal to a scalar multiple of $\vec{v}_1$? Repeat a similar process for $\vec{v}_2$.*

For parts (b) - (e), consider the following state-transition matrix:

$$\mathbf{A}=\begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

(b) Determine the eigenvalues and corresponding eigenvectors (i.e. $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

(c) Determine all of the non-zero *steady states* of the system. That is, find all possible state vectors $\vec{s}_*$ such that if Romeo and Juliet start at, or enter, any of those state vectors, their states will stay in place forever: $\{\vec{s}_* \mid \mathbf{A}\vec{s}_* = \vec{s}_*\}$.

(d) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

(e) Suppose the initial state is $\vec{s}[0]=\begin{bmatrix} 3 \\ 5 \end{bmatrix}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?
Hint: *Can you use what you learned about the eigenvectors of $\mathbf{A}$ (in parts c and d) to help you solve this problem? You can represent the starting state as a linear combination of eigenvectors $\vec{v}_1$ and $\vec{v}_2$.*

Now suppose we have the following state-transition matrix:

$$\mathbf{A}=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Use this state-transition matrix for parts (f) - (h).

(f) Determine the eigenvalues and corresponding eigenvectors (i.e. $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$) for this system. Note that this matrix is **a special case** of the matrix explored in part (a), so you can use results from that part to help you.

(g) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

(h) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Finally, we consider the case where we have the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

Use this state-transition matrix for parts (i) - (k).

(i) Determine the eigenvalues and corresponding eigenvectors (i.e. $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$) for this system. Note that this matrix is **a special case** of the matrix explored in part (a), so you can use results from that part to help you.

(j) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] = \begin{bmatrix} R[0] \\ J[0] \end{bmatrix}$, where $\vec{s}[0] \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time if $R[0] > 0$ and $J[0] < 0$? What about if $R[0] < 0$ and $J[0] > 0$? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

(k) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

5. Traffic Flows

Learning Objective: *The learning objective of this problem is to see how the concept of nullspaces can be applied to flow problems.*

Your goal is to measure the flow rates of vehicles along roads in a town. It is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by measuring the flow along only some roads. In this problem, we will explore this concept.

(a) Let's begin with a network with three intersections, A, B and C. Define the flow t_1 as the rate of cars (cars/hour) on the road between B and A, flow t_2 as the rate on the road between C and B, and flow t_3 as the rate on the road between C and A.

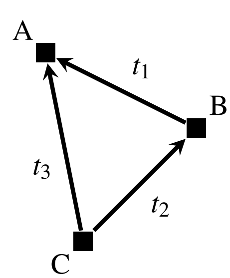

Figure 3: A simple road network.

(Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from A to C, then $t_3 = -100$. The flows now are not fractions of water in reservoirs as in the pumps setting, but numbers of cars.)

We assume the "flow conservation" constraints: the net number of cars per hour flowing into each intersection is zero. For example at intersection B, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3). If we can, find the values of t_2 and t_3.

(b) Now suppose we have a larger network, as shown in Figure 4.

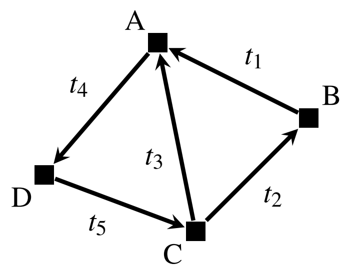

Figure 4: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads CA (measuring t_3) and DC (measuring t_5). A Stanford student claims that we need two sensors placed on the roads CB (measuring t_2) and BA (measuring t_1). Write out the system of linear equations that represents this flow graph. Is it possible to determine all traffic flows, $\left[t_1, t_2, t_3, t_4, t_5\right]^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion? *Hint: This can be solved just writing out the relevant equations and reasoning about them.*

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) i.e. the system of equations from part (b) as a matrix equation.

Construct a 4×5 matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$:

$$\begin{bmatrix} & & \mathbf{B} & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

represents the flow conservation constraints for the network in Figure 4.

*Hint: You can construct **B** using only* $0, 1,$ *and* -1 *entries. Each row represents the inflow/outflow of an intersection. This matrix is called the **incidence matrix**.*

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of all valid traffic flows for the network of Figure 4. Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix $\mathbf{B}$. That is, we can find all valid traffic flows by computing the null space of $\mathbf{B}$. What is the dimension of the nullspace?

(e) Notice that we can represent the Berkeley student's measurement as $\mathbf{M}_B\vec{t}$, where:

$$\mathbf{M}_B\vec{t} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \vec{t} = \begin{bmatrix} t_3 \\ t_5 \end{bmatrix}$$

Write a matrix $\mathbf{M}_S$ that can be used to represent the Stanford student's measurement.

(f) Now let us analyze more general road networks. Say there is a road network graph G, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a k-dimensional null space, does this mean measuring the flows along ***any*** k **roads** is always sufficient to recover all of the true flows? In other words, is there ever a possibility of being unable to recover the true flows depending on which k roads you choose?

If you think measuring the flows along any k roads will always work, then prove it showing various possible scenarios. Otherwise give an example showing a scenario where it does not work (such an example is called a counter example).

Hint: Consider the Stanford student's measurement from part (b).

(g) **[Challenge, Optional]** Assume that $\vec{u}$ and $\vec{t}$ are distinct valid flows, that is $\mathbf{B}_G\vec{u} = \mathbf{B}_G\vec{t} = \vec{0}$. Can you recover all of the network's true flows if $(\vec{u} - \vec{t})$ belongs to the nullspace of $\mathbf{M}_S$?

Clarification: A "valid" flow is one that is possible without violating the constraints on the nodes (so flow in must equal to flow out). There may be many valid flows, but only one "true" flow. The "true flow" is one of many the valid flows, which represents the actual number of cars/ hour on each road.

(h) **[Challenge, Optional]** If the incidence matrix $\mathbf{B}_G$ has a k-dimensional null space, does this mean we can **always pick a set of** k **roads** such that measuring the flows along these roads is sufficient to recover the exact flows? If this is true, explain how you would pick these k roads to guarantee that you could recover the missing information. Otherwise, give a counterexample.

6. Subspaces, Bases and Dimension

For each of the sets $\mathbb{U}$ (which are subsets of $\mathbb{R}^3$) defined below, state whether $\mathbb{U}$ is a subspace of $\mathbb{R}^3$ or not. If $\mathbb{U}$ is a subspace, find a basis for it and state the dimension. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

(a) $\mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(b) **(PRACTICE/OPTIONAL)** $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

(c) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(d) **(PRACTICE, OPTIONAL)** $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+y^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.