1. **Reading Assignment**

For this homework, please read Note 0, Note 1A, and Note 1B. This will provide an overview of linear equations and augmented matrices. You are always welcome and encouraged to read ahead as well. Write a few sentences about how the content in these notes relates to what you have learned before and what content is new.

2. **Reading Reflection**

Our modern world is filled with information devices and systems, and we want to get you thinking about them! Think about your favorite devices. If you’re stuck, here are a few examples: cell phone camera, voice-activated speaker, heart rate monitor, vocal microphone, RADAR scanner. Write a short paragraph identifying the following:

(a) How does this device physically sense and measure the real world?
(b) What does it do with the information it senses?
(c) Does this device have any actuators or other ways to respond to what it senses?
(d) What are some applications where this device is used? Are there any alternate options we can use instead of this device?
(e) What do you hope to learn about this device in this class?

We hope this inspires your learnings and curiosities in this class!

**Solution:** This question is graded purely on effort. If you have a particular curiosity, feel free to ask a TA! Many of us are doing in-depth research on these devices.

3. **Counting Solutions**

**Learning Goal:** *(This problem is designed to illustrate the different types of systems of equations. Some sub-parts will have a unique solution and others have no solutions or infinitely many solutions. In this class, we will build up the mathematical machinery to systematically determine which case applies.)*

**Directions:** For each of the following systems of linear equations, determine if there is a unique solution, no solution, or an infinite number of solutions. If there is a unique solution, find it. If there is an infinite number of solutions, explicitly state this, describe the set of all solutions, and then write one of such solutions. If there is no solution, explain why. **Show your work.**

**Example:** The below example shows how to methodically solve systems of linear equations using the substitution method.

\[
\begin{align*}
2x + 3y &= 5 \\
x + y &= 2
\end{align*}
\]

**Example Solution**

\[
\begin{align*}
2x+3y &= 5 \\
x+y &= 2
\end{align*}
\]
Subtract: $(1) - 2*(2)$

$$y = 1 \quad (3)$$

Now we plug in (3) into (2) and solve for $x$

$$x + 1 = 2$$

$$\rightarrow x = 1 \quad (4)$$

From (3) and (4), we get the unique solution:

$$x = 1$$

$$y = 1$$

(a)

$$\begin{align*}
    x + y + z &= 3 \\
    2x + 2y + 2z &= 5
\end{align*}$$

**Solution:** For each subpart, we include two solutions, one using substitution (as directed in the problem) and one using Gaussian elimination. You may give yourself credit for either approach.

**Solution A:**

$$\begin{align*}
    x + y + z &= 3 \\
    2x + 2y + 2z &= 5
\end{align*} \quad (5) \quad (6)$$

Subtract: $(6) - 2*(5)$

$$0 = -1 \quad (7)$$

We see this results in a contradiction in (7), indicating that no values of $x, y, z$ can satisfy both equations. Therefore there are no solutions.

**Solution B:**

$$\begin{bmatrix}
1 & 1 & 1 & 3 \\
2 & 2 & 2 & 5
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 3 \\
0 & 0 & 0 & -1
\end{bmatrix} \text{ using } R_2 \leftarrow R_2 - 2R_1$$

No solution. The fact that there are fewer equations than there are unknowns immediately means that it is not possible to have a unique solution; however, this does not guarantee that there is a solution to begin with. From Gaussian Elimination, we can see that these equations are contradictory since $0 \neq -1$. In other words, no values of $x, y,$ and $z$ can satisfy both equations simultaneously.

(b)

$$-y + 2z = 1$$

$$2x + z = 2$$

**Solution:**

**Solution A:**

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all.
We notice that because we cannot cancel out $x$ or $y$ using the other equation, the equations do not contradict each other so there must exist an infinite number of solutions. We choose $z$ to be our free variable and can then solve each equation in terms of $z$.

\[
x = 1 - \frac{1}{2}z
\]
\[
y = 2z - 1
\]

**Solution B:**
Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all.

\[
\begin{bmatrix}
0 & -1 & 2 \\
2 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 0 & 1 \\
0 & -1 & 2
\end{bmatrix}
\text{swapping } R_1 \text{ and } R_2
\]
\[
\rightarrow
\begin{bmatrix}
1 & 0 & \frac{1}{2} \\
0 & -1 & 1
\end{bmatrix}
\text{using } R_1 \leftarrow \frac{1}{2}R_1
\]
\[
\rightarrow
\begin{bmatrix}
1 & 0 & \frac{1}{2} \\
0 & 1 & -2
\end{bmatrix}
\text{using } R_2 \leftarrow -R_2
\]

We have now completed Gaussian elimination because we have a leading 1 in each row with zeros below that 1 in its column. In this way we can explicitly see that $z$ is a free variable ($x$ and $y$ depend on $z$ and there are no constraints on the value of $z$). Thus there are an infinite number of solutions. The set of infinite solutions has the form (for some $z \in \mathbb{R}$):

\[
x = 1 - \frac{1}{2}z
\]
\[
y = 2z - 1
\]

To get full credit it is enough to state "Infinite solutions" and give one possible solution that fits the form above.

(c)
\[
x + 2y = 5
\]
\[
2x - y = 0
\]
\[
3x + y = 5
\]

**Solution:**

**Solution A:**
In this case, there are three equations with only two unknowns. However, this fact alone does not tell us whether there is a unique solution, no solution, or an infinite number of solutions.

\[
x + 2y = 5 \quad (8)
\]
\[
2x - y = 0 \quad (9)
\]
\[
3x + y = 5 \quad (10)
\]
Adding (8) and (9), we obtain

\[ 3x + y = 5 \]  
(11)

Notice that equation (11) = equation (10)! Put another way, equation (10) provides no new information about the system that equations (8) and (9) could not tell us (importantly, it also does not contradict any information from the previous equations as well). Knowing this, we focus only on (8) and (9).

Add (8) + 2*(9)

\[ 5x = 5 \]
\[ \Rightarrow x = 1 \]  
(12)

Plugging this value of x back into (8), we obtain

\[ 1 + 2y = 5 \]
\[ \Rightarrow y = 2 \]  
(13)

Yielding the unique solution

\[ x = 1 \]
\[ y = 2 \]

Solution B:

\[
\begin{bmatrix}
1 & 2 & 5 \\
2 & -1 & 0 \\
3 & 1 & 5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 5 \\
0 & -5 & -10 \\
3 & 1 & 5
\end{bmatrix}
\]
using \( R_2 \leftarrow R_2 - 2R_1 \)

\[
\begin{bmatrix}
1 & 2 & 5 \\
0 & -5 & -10 \\
3 & 1 & 5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 5 \\
0 & 1 & 1 \\
0 & -5 & -10
\end{bmatrix}
\]
using \( R_3 \leftarrow R_3 - 3R_1 \)

\[
\begin{bmatrix}
1 & 2 & 5 \\
0 & 1 & 1 \\
0 & -5 & -10
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 5 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]
using \( R_3 \leftarrow R_3 + 5R_2 \)

\[
\begin{bmatrix}
1 & 2 & 5 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]
using \( R_1 \leftarrow R_1 - 2R_2 \)

Unique solution, \( \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)

The system of linear equations at the end of the Gaussian Elimination above simply reads out

\[ x = 1 \]
\[ y = 2 \]
\[ 0 = 0 \]
(d)

\[
\begin{align*}
  x + 2y &= 3 \\
 2x - y &= 1 \\
  x - 3y &= -5
\end{align*}
\]

**Solution:**

**Solution A:**

\[\begin{align*}
  x + 2y &= 3 \\
 2x - y &= 1 \\
  x - 3y &= -5
\end{align*}\] (14)

Add: (14) + (16)

\[\begin{align*}
  2x - y &= -2
\end{align*}\] (17)

Subtract: (15) - (17)

\[\begin{align*}
  0 &= 3
\end{align*}\]

This is a contradiction, so there is no solution.

There is no solution for this system as no choice of \(x\) and \(y\) can satisfy all equations simultaneously. This is often what happens when you have more equations than unknowns, although as you saw in the previous part, it doesn’t always happen.

**Solution B:**

\[
\begin{bmatrix}
  1 & 2 & 3 \\
  2 & -1 & 1 \\
  1 & -3 & -5
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 2 & 3 \\
  0 & -5 & -5 \\
  1 & -3 & -5
\end{bmatrix}
\]

using \(R_2 \leftarrow R_2 - 2R_1\)

\[
\begin{bmatrix}
  1 & 2 & 3 \\
  0 & -5 & -5 \\
  0 & -5 & -8
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 2 & 3 \\
  0 & 1 & 1 \\
  0 & -5 & -8
\end{bmatrix}
\]

using \(R_3 \leftarrow R_3 - R_1\)

\[
\begin{bmatrix}
  1 & 2 & 3 \\
  0 & 1 & 1 \\
  0 & -5 & -8
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 2 & 3 \\
  0 & 1 & 1 \\
  0 & 0 & -3
\end{bmatrix}
\]

using \(R_3 \leftarrow R_3 + 5R_2\)

No solution. We can think of this to mean that there are no values of \(x\) and \(y\) which satisfy the conditions in all three equations simultaneously, because in order to satisfy all three equations, the last row \(0 = -3\) would need to be true.

4. **Magic Square**

In an \(n \times n\) "magic square," all of the sums across each of the \(n\) rows, \(n\) columns, and 2 diagonals equal magic constant \(k\). For example, in the below magic square, each row, column, and diagonal sums to 34.
The magic square is a classic math puzzle, and some of you may have solved these as children by guessing. However, it turns out they can be solved systematically by setting up a system of linear equations!

(a) How many linear equations can you write for an \( n \times n \) magic square?

**Solution:**

\( 2n + 2 \), since there is one equation for each of the \( n \) rows, \( n \) columns, and 2 diagonals.

(b) For the generalized magic square below, write out a system of linear equations.

Hint: Set the sum of entries in each row, column, and diagonal equal to \( k \).

\[
\begin{array}{ccc}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{array}
\]

**Solution:**

\[
\begin{align*}
x_{11} + x_{12} + x_{13} &= k \\
x_{21} + x_{22} + x_{23} &= k \\
x_{31} + x_{32} + x_{33} &= k \\
x_{11} + x_{21} + x_{31} &= k \\
x_{12} + x_{22} + x_{32} &= k \\
x_{13} + x_{23} + x_{33} &= k \\
x_{11} + x_{22} + x_{33} &= k \\
x_{31} + x_{22} + x_{13} &= k
\end{align*}
\]

(c) Now consider the following square, with some entries filled in. Substitute the known entries into the linear equations you wrote in part (b) to solve for the missing entries \( x_{11}, x_{12}, x_{32} \). Please show the equations you use to solve; credit will not be given for solving by inspection.
<table>
<thead>
<tr>
<th></th>
<th>$x_{11}$</th>
<th>$x_{12}$</th>
<th>$x_{32}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$x_{32}$</td>
<td>6</td>
</tr>
</tbody>
</table>

**Solution:**

\[
\begin{align*}
x_{11} + x_{12} + 8 &= k \quad (18) \\
9 + 5 + 1 &= k \quad (19) \\
2 + x_{32} + 6 &= k \quad (20) \\
x_{11} + 9 + 2 &= k \quad (21) \\
x_{12} + 5 + x_{32} &= k \quad (22) \\
8 + 1 + 6 &= k \quad (23) \\
x_{11} + 5 + 6 &= k \quad (24) \\
2 + 5 + 8 &= k \quad (25)
\end{align*}
\]

From Eq. 19, $k = 15$.

Substituting $k = 15$ back into Eq. 20, $x_{32} = 7$.

Similarly, substituting $k = 15$ back into Eq. 24, $x_{11} = 4$.

Finally, substituting $k = 15, x_{32} = 7$ into Eq. 22, we find $x_{12} = 3$. 