EECS 16A Designing Information Devices and Systems I
Spring 2021 Homework 2

This homework is due February 5, 2021, at 23:59.
Self-grades are due February 8, 2021, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment

For this homework, please review Note 1B and read Note 2A. They will provide an overview of Gaussian elimination, vectors, and matrices. You are always welcome and encouraged to read beyond this as well, in particular, a quick look at Note 3 will help you. Describe how Gaussian elimination can help you understand if there are no solutions to a particular system of equations? What about a unique solution? Does a row of zeros always mean there are infinite solutions?

Solution: This is an example solution and you should give yourself full credit for any reasonable answers.

There are three situations that can result following Gaussian elimination. We assume that the system of equations has n unknowns.

Case 1: No Solution
If the augmented matrix has any rows with all-zero variable coefficients but a nonzero result (corresponding to $0 = a$ where $a \neq 0$), then there is no solution.

Case 2: Infinite Solutions
If the augmented matrix has fewer than n non-zero rows (i.e. fewer than n entries in pivot position) and any rows with all-zero variable coefficients also have a zero result (corresponding to $0 = 0$), then there are infinite solutions.

Case 3: Unique Solution
If the augmented matrix has n non-zero rows (i.e. n entries in pivot position) and any rows with all-zero variable coefficients also have a zero result (corresponding to $0 = 0$), then there is a unique solution.

2. Gaussian Elimination

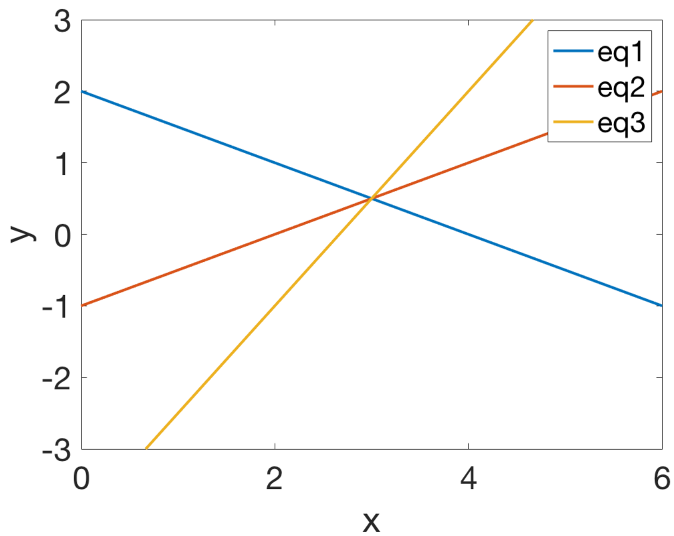
Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also practice determining the parametric solutions when there are infinitely many solutions.*

(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

i. Draw the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x + 2y = 4 \quad (1)$$
$$2x - 4y = 4 \quad (2)$$
$$3x - 2y = 8 \quad (3)$$

Solution:
The three lines intersect at the point (3,0.5).



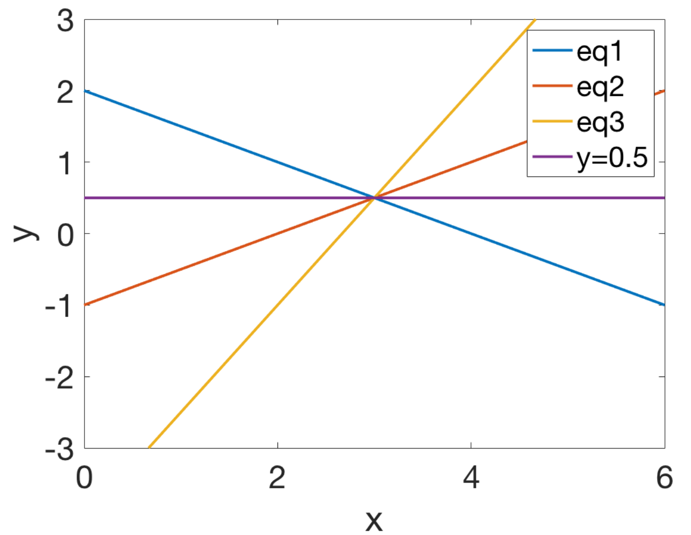
ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?
Solution: We start with the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & -4 & 4 \\ 3 & -2 & 8 \end{array}\right]$$

We then eliminate x from the second equation by subtracting $2\times$Row 1 from Row 2:

$$\text{Row 2: subtract } 2\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

So equation 2 becomes $-8y = -4$, which is equivalent to $y = 0.5$. You will notice that the line $y = 0.5$ intersects with the three lines you drew previously.



iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

Solution:

We continue from the previous part, where we had the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

and take the following steps to complete Gaussian elimination:

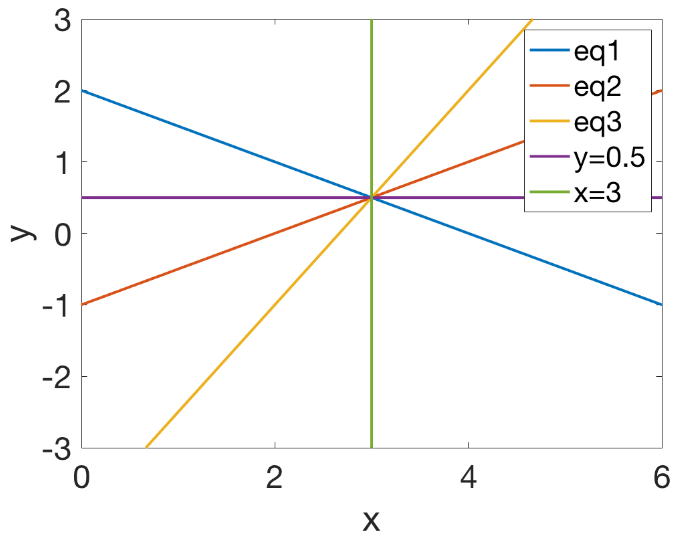
$$\text{Row 3: subtract } 3\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 0 & -8 & -4 \end{array}\right]$$

$$\text{Row 2: divide by } -8 \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & -8 & -4 \end{array}\right]$$

$$\text{Row 3: subtract } -8\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, we end up with the solution $x = 3$ and $y = 0.5$.
Plotting the new equation $x = 3$ on the same graph as before, we see that all five lines intersect at the same point (3,0.5).



(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination.

$$\begin{aligned} x+2y+5z &= 3 \\ x+12y+6z &= 1 \\ 2y+z &= 4 \\ 3x+16y+16z &= 7 \end{aligned}$$

Solution:
Writing the system in augmented matrix form we get the following:

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 1 & 12 & 6 & 1 \\ 0 & 2 & 1 & 4 \\ 3 & 16 & 16 & 7 \end{array}\right]$$

We eliminate the x variables from the second and fourth equations:

$$\begin{array}{c} \text{Row 2: subtract Row 1} \\ \text{Row 4: subtract } 3\times\text{Row 1} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 10 & 1 & -2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

We then divide Row 2 by 10 to get a 1 in the pivot position:

$$\text{Row 2: divide by 10} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

Next, we eliminate the y variables from the third and fourth equations:

$$\begin{array}{c}\text{Row 3: subtract } 2\times\text{Row 2}\\ \text{Row 4: subtract } 10\times\text{Row 2}\end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 0.8 & 4.4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We divide Row 3 by 0.8 to get a 1 in the pivot position:

$$\text{Row 3: divide by } 0.8 \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We then proceed with back-substitution:

$$\begin{array}{c}\text{Row 2: subtract } 0.1\times\text{Row 3}\\ \text{Row 1: subtract } 5\times\text{Row 3}\end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 0 & -24.5 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This final matrix is in reduced row echelon form. The first three rows of the matrix have non-zero elements in pivot position, for a system with three unknowns, and the fourth row is a row of zeros, so we can conclude there is a unique solution: $x = -23$, $y = -0.75$, and $z = 5.5$.

(c) Consider the following system of equations:

$$\begin{aligned} x + 2y + 5z &= 6 \\ 3x + 9y + 6z &= 3 \end{aligned}$$

You are given a set S of candidate solutions,

$$S = \left\{ \vec{v} \;\middle|\; \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} -11 \\ 3 \\ 1 \end{bmatrix} t \quad , \quad t \in \mathbb{R} \right\}$$

This vector notation can be expressed in terms of its components:

$$\vec{v} = \begin{bmatrix} 16 - 11t \\ -5 + 3t \\ t \end{bmatrix} \quad \text{means} \quad \begin{aligned} x &= 16 - 11t \\ y &= -5 + 3t \\ z &= t \end{aligned}$$

Show, by substitution, that any $\vec{v} \in S$ is a solution to the system of equations given above. Note that this means that the candidate solution must satisfy the system of equations for all $t \in \mathbb{R}$.

Solution:

We can use direct substitution to show any $\vec{v} \in S$ is a solution to the equations.

$$\vec{v} = \begin{bmatrix} 16-11t \\ -5+3t \\ t \end{bmatrix} \implies \begin{aligned} x &= 16-11t \\ y &= -5+3t \\ z &= t \end{aligned}$$

We now take these expressions for x, y, and z, and show that the system of equations is satisfied for all $t \in \mathbb{R}$.

Plugging the expressions into equation 1 yields the following:

$$\begin{aligned} (16-11t)+2(-5+3t)+5t &= 6 \\ 16-11t-10+6t+5t &= 6 \\ 6 &= 6 \end{aligned}$$

This holds for all $t \in \mathbb{R}$.

Similarly, plugging the expressions into equation 2 yields the following:

$$\begin{aligned} 3(16-11t)+9(-5+3t)+6t &= 3 \\ 48-33t-45+27t+6t &= 3 \\ 3 &= 3 \end{aligned}$$

This also holds for all $t \in \mathbb{R}$.

Therefore, the candidate solution satisfies the system of equations for all $t \in \mathbb{R}$.

(d) Consider the following system:

$$\begin{aligned} 4x+4y+4z+w+v &= 1 \\ x+y+2z+4w+v &= 2 \\ 5x+5y+5z+w+v &= 0 \end{aligned}$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Determine the solutions to the set of equations.

Solution:

We first note that the given augmented matrix is in reduced row echelon form, which makes sense as it is the final output of the Gaussian elimination algorithm. We observe that the second and fifth columns do not have 1s in pivot position so there are two free variables corresponding to y and v.

Let $y = s$ and let $v = t$, where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.
Using back substitution, we can solve for x, y, z, w, and v in terms of s and t:

$$\begin{aligned} &\text{Row 1:} \quad x+y+3v=16 \implies x = 16-3t-s \\ &\text{Row 2:} \quad z-3v=-17 \implies z=-17+3t \\ &\text{Row 3:} \quad w+v=5 \implies w=5-t \end{aligned}$$

The solutions to the system of equations are therefore:

$$\begin{aligned} x &= 16-3t-s \\ y &= s \\ z &= -17+3t \\ w &= 5-t \\ v &= t \end{aligned}$$

The solutions can also be represented by a set as:

$$S = \left\{ \vec{u} \;\middle|\; \vec{u} = \begin{bmatrix} 16 \\ 0 \\ -17 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} t \quad , \quad s \in \mathbb{R}, \quad t \in \mathbb{R} \right\}.$$

3. Linear Dependence

Learning Goal: *Evaluate the linear dependency of a set of vectors.*

State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

(a) $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

Solution: The vectors are linearly independent. A set of two vectors can only be linearly dependent if one of the vectors is a scaled version of the other. $\begin{bmatrix} -5 \\ 2 \end{bmatrix} \neq \alpha \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ no matter the chosen $\alpha \in \mathbb{R}$.

(b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Solution: This set of vectors is linearly dependent. To find a linear combination that yields $\vec{0}$, find the RREF of the following augmented matrix:

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow -R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 + R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow R_2/3]{R_5 \leftarrow R_5 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 + 3R_2]{R_3 \leftarrow R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Choosing the value of the free variable as 3α (chosen to make the coefficients nice) gives the following linear combination that shows linear dependence:

$$-2\alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix} + 3\alpha \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If you have a set of coefficients that match a specific value of α, give yourself full credit.

(c) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution: This set of vectors is linearly dependent. Since the subset of the first 3 vectors is linearly independent, they span $\mathbb{R}^3$. Since the fourth vector is in $\mathbb{R}^3$, we are guaranteed some linear combination of the first three vectors that yields the fourth. To find a specific linear combination that shows linear dependence find the RREF of the following augmented matrix:

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 0 & 0 \\ 2 & 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1/2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -3 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3/-3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right] \xrightarrow[R_1 \leftarrow R_1 - R_3]{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right]$$

Choosing the value of the free variable as 3α (chosen to make the coefficients nice) gives the following linear combination that shows linear dependence:

$$-2\alpha \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\alpha \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + 3\alpha \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If you have a set of coefficients that match a specific value of α, give yourself full credit.

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Solution: Since this set contains $\vec{0}$, it is linearly dependent as we can take the linear combination $\alpha \cdot \vec{0}$ where $\alpha \neq 0$ to get $\vec{0}$.

4. Filtering Out The Troll

Solution: #SystemsOfEquations #LinearCombination

Learning Goal: *The goal of this problem is to represent a practical scenario using a simple model of directional microphones. You will tackle the problem of sound reconstruction through solving a system of linear equations.*

You attended a very important public speech and recorded it using a recording device that had two directional microphones. However, there was a person in the audience who was trolling around, adding interference to the recording. When you went back home to listen to the recording, you realized that the two recordings were dominated by the troll's interference and you could not hear the speech. Fortunately, since your recording device contained two microphones, you realized there is a way to combine the two individual microphone recordings so that the troll's interference is removed. You remembered the locations of the speaker and the troll and created the diagram shown in Figure 1. You (and your two microphones) are located at the origin.

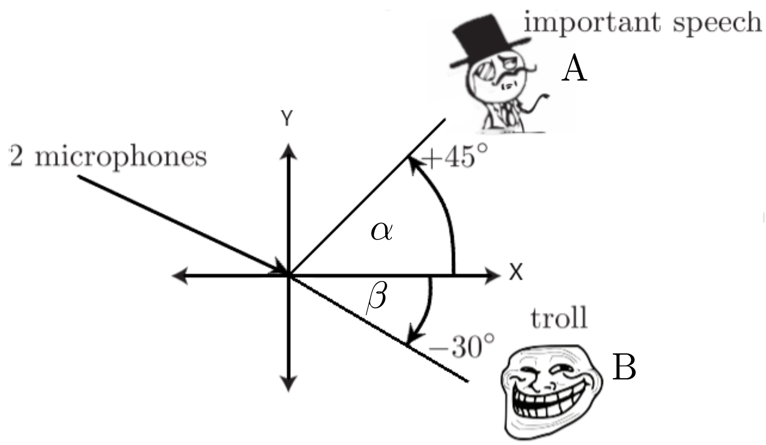


Figure 1: Locations of the speaker and the troll.

Each directional microphone records signals differently based on their angles of arrival. The first microphone weights (multiplies) a signal, coming from an angle θ with respect to the x-axis, by the factor $f_1(\theta) = \cos(\theta)$. If two signals are simultaneously playing (as is the case with the speech and the troll interference), then a linear combination (i.e. a weighted sum) of both the signals is recorded, each weighted by the respective $f_1(\theta)$ for their angles. The second microphone weights a signal, coming from an angle θ with respect to the x-axis, by the factor $f_2(\theta) = \sin(\theta)$. Again, if there are two simultaneous signals, then the second microphone also records the weighted sum of both the signals.

For example, an audio source that lies on the x axis will be recorded by the first microphone with weight equal to 1 (since $\cos(0) = 1$), but will not be recorded up by the second microphone (since $\sin(0) = 0$). Note that the weights can also be negative.

Let us represent the speech sample at a particular time-instant by the variable a and the interference caused by the troll at the same time-instant by the variable b. Remember, we do not know either a or b. The recording of the first microphone at that time instant is given by m_1:

$$m_1 = f_1(\alpha) \cdot a + f_1(\beta) \cdot b,$$

and the second microphone recorded the signal

$$m_2 = f_2(\alpha) \cdot a + f_2(\beta) \cdot b.$$

where α and β are the angles at which the public speaker A and the troll B respectively are located with respect to the x-axis, and variables a and b are the audio signals produced by the public speaker A and the troll B respectively.

(As a side note, we could represent the entire speech with a vector $\vec{a}$ by stacking all the speech samples on top of each other, and this is what we would typically do in a real-world speech processing example. However, here we consider just one time instant of the speech for simplicity.)

(a) Plug in the values of α and β to write the recordings of the two microphones m_1 and m_2 as a linear combination (i.e. a weighted sum) of a and b.

Solution:

$$
\begin{aligned}
m_1 &= \cos\left(\frac{\pi}{4}\right)\cdot a + \cos\left(-\frac{\pi}{6}\right)\cdot b \\
&= \frac{1}{\sqrt{2}}\cdot a + \frac{\sqrt{3}}{2}\cdot b \\
m_2 &= \sin\left(\frac{\pi}{4}\right)\cdot a + \sin\left(-\frac{\pi}{6}\right)\cdot b \\
&= \frac{1}{\sqrt{2}}\cdot a - \frac{1}{2}\cdot b
\end{aligned}
$$

(b) Solve the system you wrote out on the earlier part to recover the important speech a, as a weighted combination of m_1 and m_2. In other words, write $a = u\cdot m_1 + v\cdot m_2$ (where u and v are scalars). What are the values of u and v?

Solution: Solving the system of linear equations yields

$$
a = \frac{\sqrt{2}}{1+\sqrt{3}}\cdot\left(m_1 + \sqrt{3}m_2\right).
$$

Therefore, the values are $u = \frac{\sqrt{2}}{1+\sqrt{3}}$ and $v = \frac{\sqrt{6}}{1+\sqrt{3}}$.

It is fine if you solved this either using IPython or by hand using any valid technique. The easiest approach is to subtract either of the two equations from the other and immediately see that $b = \frac{2}{\sqrt{3}+1}(m_1 - m_2)$. Substituting b back into the second equation and multiplying through by $\sqrt{2}$ gives that $a = \sqrt{2}(m_2 + \frac{1}{\sqrt{3}+1}(m_1 - m_2))$, which simplifies to the expression given above.

Notice that subtracting one equation from the other is natural given the symmetry of the microphone patterns and the fact that the patterns intersect at the 45 degree line where the important speech is happening, and the fact that $\sin(45^\circ) = \cos(45^\circ)$. So we know that the result of subtracting one microphone recording from the other results in only the troll's contribution. Once we have the troll contribution, we can remove it and obtain the important speaker's sole content.

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

Solution:

The solution code can be found in `sol2.ipynb`. The speaker says: "All human beings are born free and equal in dignity and rights." and the speech was taken from the Universal Declaration of Human Rights.

The idea of using multiple microphones to isolate speech is interesting and is increasingly used in practice. Furthermore, similar techniques are used in wireless communication both by cellular systems like LTE and increasingly by WiFi hotspots. (This is why they often have multiple antennas).

5. [PRACTICE/OPTIONAL] Tyler's Optimal Tea

Solution: `#SystemsOfEquations #GaussianElimination`

Learning Goal: *Recognize a problem that can be cast as a system of linear equations.*

Tyler's Optimal Tea has a unique way of serving its customers. To ensure the best customer experience, each customer gets a combination drink personalized to their tastes. Tyler knows that a lot of customers don't know what they want, so when customers walk up to the counter, they are asked to taste four standard combination drinks that each contain a different mixture of the available pure teas.

Each combination drink (Classic, Roasted, Mountain, and Okinawa) is made of a mixture of pure teas (Black, Oolong, Green, and Earl Grey), with the total amount of pure tea in each combination drink always the same, and equal to one cup. The table below shows the quantity of each pure tea (Black, Oolong, Green, and Earl Grey) contained in each of the four standard combination drinks (Classic, Roasted, Mountain, and Okinawa).

Tea [cups]	Classic	Roasted	Mountain	Okinawa
Black	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$
Oolong	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{3}$
Green	0	$\frac{1}{3}$	$\frac{3}{5}$	0
Earl Grey	$\frac{1}{3}$	0	0	0

Initially, the customer's ratings for each of the pure teas are unknown. Tyler's goal is to determine how much the customer likes each of the pure teas, so that an optimal combination drink can then be made. By letting the customer taste and score each of the four standard combination drinks, Tyler can use linear algebra to determine the customer's initially unknown ratings for each of the pure teas. After a customer gives a score (all of the scores are real numbers) for each of the four standard combination drinks, Tyler then calculates how much the customer likes each pure tea and mixes up a special combination drink that will maximize the customer's score.

The score that a customer gives for a combination drink is a linear combination of the ratings of the constituent pure teas, based on their proportion. For example, if a customer's rating for black tea is 6 and oolong tea is 3, then the total score for the Okinawa Tea drink would be $6 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = 5$ because Okinawa has $\frac{2}{3}$ black tea and $\frac{1}{3}$ Oolong tea.

Professor Waller was thirsty after giving the first lecture, so Professor Waller decided to take a drink break at Tyler's Optimal Tea. Professor Waller walked in and gave the following ratings:

Combination Drink	**Score**
Classic	7
Roasted	7
Mountain	$7\frac{2}{5}$
Okinawa	$6\frac{1}{3}$

(a) What were Professor Waller's ratings for each tea? **Work this problem out by hand in terms of the steps. You may use a calculator to do algebra.**

Solution:

Using Professor Waller's ratings, Tyler mentally records the following system of equations. Let x_b be the customer's rating of black tea, x_o be the customer's rating of oolong tea, x_g be the customer's rating of green tea, and x_e be the customer's rating of earl grey tea.

$$\begin{aligned}
\text{Classic:} \quad & 7 = \frac{1}{3}x_\text{b} + \frac{1}{3}x_\text{o} + \frac{1}{3}x_\text{e} \\
\text{Roasted:} \quad & 7 = \frac{1}{3}x_\text{b} + \frac{1}{3}x_\text{o} + \frac{1}{3}x_\text{g} \\
\text{Mountain:} \quad & 7\frac{2}{5} = \frac{2}{5}x_\text{o} + \frac{3}{5}x_\text{g} \\
\text{Okinawa:} \quad & 6\frac{1}{3} = \frac{2}{3}x_\text{b} + \frac{1}{3}x_\text{o}
\end{aligned}$$

Multiply each equation by the denominator of the fraction (in order to make them easier to read):

$$\begin{aligned}
21 &= x_\text{b} + x_\text{o} + x_\text{e} \\
21 &= x_\text{b} + x_\text{o} + x_\text{g} \\
37 &= 2x_\text{o} + 3x_\text{g} \\
19 &= 2x_\text{b} + x_\text{o}
\end{aligned}$$

The above equations can be written as an augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 1 & 1 & 1 & 0 & 21 \\ 0 & 2 & 3 & 0 & 37 \\ 2 & 1 & 0 & 0 & 19 \end{array}\right].$$

Row reduce the matrix into reduced row echelon form as follows. (It's fine if you solved the system of equations by hand a different way. Here, however, we will demonstrate how to do it using Gaussian elimination.)

Noting that there is a 1 in the upper left hand corner, subtract Row 1 from Row 2 and 2×Row 1 from Row 4.

$$\begin{array}{c}\text{Row 2: subtract Row 1} \\ \text{Row 4: subtract } 2\times\text{Row 1}\end{array} \implies \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 & 37 \\ 0 & -1 & 0 & -2 & -23 \end{array}\right]$$

Since Row 2 has a 0 in the diagonal element, multiply Row 4 by -1 and then switch Rows 2 and 4.

$$\begin{array}{c}\text{Multiply Row 4 by } -1 \\ \text{Switch Row 2 and Row 4}\end{array} \implies \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 2 & 3 & 0 & 37 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Subtract Row 2 from Row 1 and 2×Row 2 from Row 3.

$$\begin{array}{c}\text{Row 1: subtract Row 2}\\ \text{Row 3: subtract 2×Row 2}\end{array} \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 0 & 3 & -4 & -9 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Switch Row 3 and Row 4 and subtract 3× the new Row 3 from the new Row 4.

$$\begin{array}{c}\text{Switch Row 3 and Row 4}\\ \text{Row 4: subtract 3×Row 3}\end{array} \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -9 \end{array}\right]$$

Finally, multiply Row 4 by -1 and add Row 4 to Row 1 and Row 3 and subtract 2×Row 4 from Row 2.

$$\begin{array}{c}\text{Multiply Row 4 by } -1\\ \text{Row 1, Row 3: add Row 4}\\ \text{Row 2: subtract 2×Row 4}\end{array} \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

Professor Waller's ratings for each tea are

Tea	Score
Black	7
Oolong	5
Green	9
Earl Grey	9

(b) What mystery tea combination could Tyler put in Professor Waller's personalized drink to maximize the customer's score? If there is more than one correct answer, state that there are many answers, and give one such combination. What score would Professor Waller give for the answer you wrote down? Assume the total amount of tea must be one cup.

Solution:

There are many answers. *Any* combination of green tea and earl grey is acceptable as they have equal ratings. More precisely, for any $0 \leq a \leq 1$, a combination with a cups of green tea and $1-a$ cups of earl grey will yield a score of $9a + 9(1-a) = 9$.

As an example, Tyler could choose $a = \frac{1}{2}$ so that Professor Waller's drink has $\frac{1}{2}$ cup of green tea and $\frac{1}{2}$ cup of earl grey.

How to see this? Green tea and earl grey are tied for Professor Waller's favorite tea, so it doesn't make a difference if Tyler substitutes one for the other in any quantity - the score remains 9. It also doesn't make sense to substitute a less preferred tea like black tea for Professor Waller's favorite tea, or to add a less preferred tea at the expense of the most preferred teas, as this will lead to scores less than 9.

6. Fountain Codes

Learning Goal: *Linear algebra shows up in many important engineering applications. Wireless communication and information theory heavily rely on principles of linear algebra. This problem illustrates some of the techniques used in wireless communication.*

Alice wants to send a message to her friend Bob. Alice sends her message $\vec{m}$ across a wireless channel in the form of a transmission vector $\vec{w}$. Bob receives a vector of symbols denoted as $\vec{r}$.

Alice knows some of the symbols in the transmission vector that she sends may be corrupted, so she needs a way to protect her message from the corruptions. (Transmission corruptions occur commonly in real wireless communication systems, for instance, when your laptop connects to a WiFi router, or your cellphone connects to the nearest cell tower.) Ideally, Alice can come up with a transmission vector such that if some of the symbols get corrupted, Bob can still figure out what Alice is trying to say!

One way to accomplish this goal is to use fountain codes, which are part of a broader family of codes called error correcting codes. The basic principle is that instead of sending the exact message, Alice sends a modified longer version of the message so that even if some parts are corrupted Bob can recover what she meant. Fountain codes are based on principles of linear algebra, and were actually developed right here at Berkeley! The company that commercialized them, Digital Fountain, (started by a **Berkeley grad, Mike Luby**), was later acquired by Qualcomm. In this problem, we will explore some of the underlying principles that make fountain codes work in a very simplified setting.

The message that Alice wants to send to Bob is the three numbers a, b, and c. The message vector representing these numbers is $\vec{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Figure 2 shows how Alice's message is encoded in a transmission vector and how Bob's received vector may have some corrupted symbols.

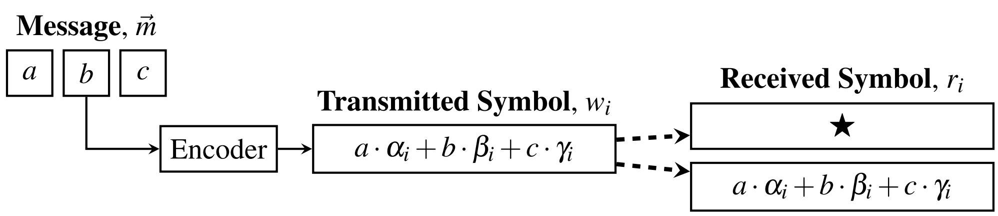


Figure 2: Each symbol in a transmission vector $\vec{w}$ is a linear combination of a, b, and c. Each transmitted symbol, w_i, is either received exactly as it was sent, or it is corrupted. A corrupted symbol is denoted by ★. The ith row of a symbol generating matrix G determines the values of α_i, β_i, and γ_i.

(a) Since Alice has three numbers she wishes to send, she could transmit six symbols in her transmission vector for redundancy. This transmission strategy is called the "repetition code".

If Alice uses the repetition code, her transmission vector is $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$.

As depicted in Figure 2, the received vector may have corrupted symbols. For example, suppose only the first symbol was corrupted, then Bob would receive the vector $\vec{r} = \begin{bmatrix} \star \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$, where the ★ symbol represents a corrupted symbol.

Using the repetition code scheme, give an example of a received vector $\vec{r}$ with only two corrupted symbols such that a is unrecoverable but b and c are still recoverable.

Solution:

If Bob receives the vector $\vec{r} = \begin{bmatrix} \star \\ b \\ c \\ \star \\ b \\ c \end{bmatrix}$ then there is no way for him to recover a.

(b) Alice can generate $\vec{w}$ by multiplying her message $\vec{m}$ by a matrix. Write a matrix-vector multiplication that Alice can use to generate $\vec{w}$ according to the repetition code scheme. Specifically, find a "generating" matrix G_R such that $G_R\vec{m} = \vec{w}$, where $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$.

Solution:

Alice can use the following symbol generating matrix G_R to generate $\vec{w}$:

$$G_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Instead of a repetition code, it is also possible to use other codes (e.g. fountain codes). Alice and Bob can choose any symbol generating matrix, as long as they agree upon it in advance. Each different matrix represents a different "code." Alice's TA recommends using the symbol generating matrix G_F:

$$G_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Alice then uses the symbol generating matrix G_F to produce a new transmission vector: $G_F\vec{m} = \vec{w}$.

Suppose Bob receives the vector $\vec{r} = \begin{bmatrix} 7 \\ \star \\ \star \\ 3 \\ 4 \\ \star \\ \star \end{bmatrix}$, which is a corrupted version of $\vec{w}$.

Write a system of linear equations that Bob can use to recover the message vector $\vec{m}$. Solve it to recover the three numbers that Alice sent.

Hint: Consider the rows of G_F that correspond to the uncorrupted symbols in $\vec{r}$.

Solution:

In order to recover Alice's message $\vec{m}$, Bob needs to solve the following equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \vec{m} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 4 \end{bmatrix}$$

Bob can solve this system by writing the system in augmented matrix form and using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{l} \text{Row 2: subtract Row 1} \\ \text{Row 3: subtract Row 1} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

So Bob recovers the message:

$$\vec{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ -3 \end{bmatrix}$$

(d) We explore one example case where Alice and Bob agree to use G_F (i.e. $G_F \vec{m} = \vec{w}$) and there are three corruptions, so Bob receives four uncorrupted symbols.

Suppose Bob receives $\vec{r} = \begin{bmatrix} 1 \\ \bigstar \\ 3 \\ \bigstar \\ 4 \\ \bigstar \\ 9 \end{bmatrix}$. Can you determine the message $\vec{m}$ that Alice sent?

Solution: If Bob receives the odd symbols, this corresponds to the following system of equations:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vec{m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 9 \end{bmatrix}$$

This can be expressed as an augmented matrix and solved using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 9 \end{array}\right]$$

Row 3: subtract Row 1
Row 4: subtract Row 1 $\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & 8 \end{array}\right]$

Swap Row 2 and Row 4 $\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Row 2: subtract Row 3
Row 4: subtract Row 3 $\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So Bob recovers the message:

$$\vec{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

Note: it can be shown that receiving *any* four uncorrupted symbols when Alice is using G_F is enough to recover Alice's message. On the other hand, we showed in part (a) that receiving *any* four uncorrupted symbols using G_R does not guarantee we can recover Alice's message. This is why, in practice, we would prefer to use the fountain code G_F instead of the repetition code G_R — it is a more reliable way to send messages.

7. Show It!

Learning Goal: *This is an opportunity to practice your proof development skills. For proofs you have seen before, use this as an opportunity to make sure you understood them by trying them independently using the steps discussed in class.*

(a) **Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions, then columns of A are linearly dependent.**

This problem has 4 sub-parts and the following is a chart showing the sequential steps we are going to take to approach this proof.

In a textbook you might see the steps in a proof written out in the order in the middle column of the table. But when you are building a proof you usually want to go in another order — this is the order of the subparts in this problem.

(i) **First Step: write what you know**

Think about the *information we already know* from the problem statement. We know that system of equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions. Infinitely many solutions are hard to work with, but perhaps we can simplify to something that we can work with. If the system has infinite number of solutions, it must have at least ___ distinct solutions (Fill in the blank).

Proof steps		Corresponding problem sub-parts
1	Write what is known	Sub-part (i)
2	Manipulate what is known	Sub-part (iii)
3	Connecting it up	Sub-part (iv)
4	What is to be shown	Sub-part (ii)

So let us assume that $\vec{u}$ and $\vec{v}$ are two different vectors, both of which are solutions to $\mathbf{A}\vec{x} = \vec{b}$. Express the sentence above in a mathematical form (Just writing the equations will suffice; no need to do further mathematical manipulation).

Solution: If the system has infinite number of solutions, it must have at least two distinct solutions.

(Self-grading comment: Do not reduce points if you forgot to write the answer to the blank in your solutions)

$\vec{u}$ and $\vec{v}$ must satisfy:

$$\mathbf{A}\vec{u} = \vec{b}, \quad \mathbf{A}\vec{v} = \vec{b}. \tag{4}$$

$$\vec{u} \neq \vec{v}. \tag{5}$$

(ii) **What we want to show:**

Now consider *what we need to show*. We have to show that the columns of $\mathbf{A}$ are linearly dependent. Let us assume that $\mathbf{A}$ has columns $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix}$. Using the definition of linear dependence from **Note 3 Subsection 3.1.1**, write a mathematical equation that conveys linear dependence of $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$.

Solution: According to the definition of linear dependence:

$$\alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \ldots + \alpha_n\vec{c}_n = \vec{0}. \tag{6}$$

where not all α_i's are equal to zero.

(iii) **Manipulating what we know:**

Now let us try to start from the **First step: equations from (i)**, make mathematically logical steps and reach the **What we want to show: equations from (ii)**. Since your answer to (ii) is expressed in terms of the column vectors of $\mathbf{A}$, let us try to express the mathematical equations from (i), in terms of the column vectors too. For example, we can write

$$\begin{aligned} \mathbf{A}\vec{x} &= \vec{b} \\ \implies \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} &= \vec{b} \\ \implies x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n &= \vec{b} \end{aligned}$$

Notice that x_1,... x_n etc are scalars. Now use your answer to part (i) to repeat the above formulation for distinct solutions $\vec{u}$ and $\vec{v}$.

Solution:

$$
\begin{aligned}
\mathbf{A}\vec{u} &= \vec{b} \\
\implies \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} &= \vec{b} \\
\implies u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n\vec{c}_n &= \vec{b}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}\vec{v} &= \vec{b} \\
\implies \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} &= \vec{b} \\
\implies v_1\vec{c}_1 + v_2\vec{c}_2 + \ldots + v_n\vec{c}_n &= \vec{b}
\end{aligned}
$$

(iv) **Connecting it up:**
Now think about how you can mathematically manipulate your answer from part (iii) (**Manipulating what we know**) to **match the pattern** of your answer from part (ii) (**What we want to show**).
Solution: Subtracting the second equation from the first equation in part (iii), we have

$$u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n - v_1\vec{c}_1 - v_2\vec{c}_2 - \ldots - v_n\vec{c}_n = \vec{b} - \vec{b} \quad (7)$$

$$\implies (u_1 - v_1)\vec{c}_1 + (u_2 - v_2)\vec{c}_2 + \ldots + (u_n - v_n)\vec{c}_n = \vec{0} \quad (8)$$

Let $\alpha_1 = u_1 - v_1$, ..., and $\alpha_n = u_n - v_n$, i.e. $\vec{\alpha} = \vec{u} - \vec{v}$. Here not all α_i's are equal to zero since $\vec{u} \neq \vec{v}$. Hence the mathematical expression from part (ii) (the **Final Step**) is satisfied, i.e. the proof is complete!

(b) Now try this proof on your own. Similar proofs will also be covered in your discussion section 2A. Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belongs in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

In summary, you have to prove the problem statement from both directions.

Solution:
Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n$$

We can change the scalar values to adjust for the combined vectors. Thus, we have shown that $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.

(c) Let n be a positive integer. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of k linearly dependent vectors in $\mathbb{R}^n$. Show that for *any* $n \times n$ matrix $\mathbf{A}$, the set $\{\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2, \ldots, \mathbf{A}\vec{v}_k\}$ is a set of linearly dependent vectors.

Solution:

The definition of a set of linear dependent vectors states that there exist k scalars, $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ in $\mathbb{R}$ that are *not all equal to zero* (or equivalently *at least one of which is not zero*), such that

$$\alpha_1 \cdot \vec{v}_1 + \alpha_2 \cdot \vec{v}_2 + \cdots + \alpha_k \cdot \vec{v}_k = \vec{0}. \tag{9}$$

By left-multiplying Equation 9 with $\mathbf{A}$, we get

$$\mathbf{A}\left(\alpha_1 \cdot \vec{v}_1 + \alpha_2 \cdot \vec{v}_2 + \cdots + \alpha_k \cdot \vec{v}_k\right) = \mathbf{A}\vec{0}.$$

Note that $\mathbf{A}\vec{0} = \vec{0}$. By distributing $\mathbf{A}$, we get

$$\mathbf{A}\left(\alpha_1 \cdot \vec{v}_1\right) + \mathbf{A}\left(\alpha_2 \cdot \vec{v}_2\right) + \cdots + \mathbf{A}\left(\alpha_k \cdot \vec{v}_k\right) = \vec{0}.$$

Use associativity of multiplication

$$\left(\mathbf{A}\alpha_1\right)\vec{v}_1 + \left(\mathbf{A}\alpha_2\right)\vec{v}_2 + \cdots + \left(\mathbf{A}\alpha_k\right)\vec{v}_k = \vec{0}.$$

Since scalar-matrix multiplication is commutative, we can move the scalar α in front of the matrix $\mathbf{A}$

$$\left(\alpha_1\mathbf{A}\right)\vec{v}_1 + \left(\alpha_2\mathbf{A}\right)\vec{v}_2 + \cdots + \left(\alpha_k\mathbf{A}\right)\vec{v}_k = \vec{0}.$$

Use associativity of multiplication again

$$\alpha_1 \cdot (\mathbf{A}\vec{v}_1) + \alpha_2 \cdot (\mathbf{A}\vec{v}_2) + \cdots + \alpha_k \cdot (\mathbf{A}\vec{v}_k) = \vec{0}. \tag{10}$$

Because the same k scalars, $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, are used the vectors $\{\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2, \ldots, \mathbf{A}\vec{v}_k\}$ are linear dependent.

■

To better understand the proof, here is a numerical example. Assume that we have 4 vectors,

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

In order to show that these 4 vectors are linearly dependent, we need at least one non-zero scalar that makes the linear combination of these vectors add to zero.

$$\alpha_1 \cdot \vec{v}_1 + \alpha_2 \cdot \vec{v}_2 + \alpha_3 \cdot \vec{v}_3 + \alpha_4 \cdot \vec{v}_4 = \vec{0}.$$

Here are two example sets of scalars in which at least one scalar is not equal to zero. Therefore, we have shown that the set of vectors, $\vec{v}_1 \ldots \vec{v}_4$, are linearly dependent.

$$\alpha_1 = -2, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 1.$$
$$\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 1, \alpha_4 = 0.$$

Now we want to see whether or not the set of vectors, $\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2, \mathbf{A}\vec{v}_3, \mathbf{A}\vec{v}_4$, are also linearly dependent. Note, the matrix $\mathbf{A}$ can be any 2×2 matrix.

From the proof above, we know that the same scalars, $\alpha_1 \dots \alpha_4$, will make the linear combination of the vectors add to zero. But let's see if that happens in our specific example.

Let's choose the matrix $\mathbf{A}$ and show that the set $\alpha_1 = -2, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 1$ does in fact give us a linear combination that adds to zero.

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{A}\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{A}\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{A}\vec{v}_4 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\alpha_1 \cdot (\mathbf{A}\vec{v}_1) + \alpha_2 \cdot (\mathbf{A}\vec{v}_2) + \alpha_3 \cdot (\mathbf{A}\vec{v}_3) + \alpha_4 \cdot (\mathbf{A}\vec{v}_4) =$$

$$-2 \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Great! We have now shown that the set of scalars α that made the linear combination of vectors $\vec{v}_i$ add to zero also made the linear combination of vectors $\mathbf{A}\vec{v}_i$ add to zero. You can change the numbers of the matrix $\mathbf{A}$ and everything still works. A numerical example like this is a great place to start if you are ever struggling with a proof.

Note: There are alternative and equivalent implications of linear dependence that can be used in proof (instead of Equation 9). For example:

> Some vector $\vec{v}_j$ can be represented as a linear combination of the *other* vectors as follows: There exist scalars $\alpha_i, 1 \leq i \leq k, i \neq j$, such that
>
> $$\sum_{\substack{i=1 \\ i \neq j}}^{k} \alpha_i \cdot \vec{v}_i = \vec{v}_j.$$
>
> (In this alternative, the scalars may all be zeros, and the linear combination on the left hand side must exclude $\vec{v}_j$.)

For alternatives, the rest of the proof is similar to the one demonstrated above (multiply both sides by $\mathbf{A}$ and then include the $\mathbf{A}$ in the relevant sum and next to the $\vec{v}_i$) and will result in an equation similar to Equation 10.

Common mistakes included:

- When using the definition provided in Equation 9, not indicating that at least one of scalars needs to be non-zero.
- When using the definition provided in Equation 9, stating that all scalars need to be non-zero.
- When using the alternative implication above, requiring that at least one scalar be non-zero (or all non-zero).
- When using the alternative implication above, not excluding the vector on the right hand side from the linear combination on the left hand side.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.

