EECS 16A Designing Information Devices and Systems I
Spring 2021 Homework 5

This homework is due Friday, February 26, 2021, at 23:59.
Self-grades are due Tuesday, March 2, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.
- **We strongly recommended that you submit your self-grades PRIOR to taking Midterm 1 on March 1, 2021, since looking at the solutions earlier will help you to study for the midterm.**

Submit the file to the appropriate assignment on Gradescope.

1. Reading Assignment

For this homework, please read Note 8 through 9. These notes will give you an overview of matrix subspaces and eigenvalues/eigenvectors. You are always welcome and encouraged to read beyond this as well.

2. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

Self-grading note: For this subproblem and the following subproblems which involve computing eigenvectors, give yourself full credit if the eigenvector(s) you calculated is/are a scaled (i.e, multiplied by a real valued α) version of the eigenvector(s) given in the solutions.

There are two ways to do this.

First, we can do it by inspection. We can see that this matrix multiplies everything in the first coordinate by 5 and everything in the second by 2. Consequently, when given $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it will return 2 times the input. And when given $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it will return 5 times the input vector.

Alternatively, we can use determinants.

$$\det\left(\begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(5-\lambda)(2-\lambda)-0=0$$

This is already factored for you! We see that, by definition, diagonal matrices have their eigenvalues on the diagonal.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A}-5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix}5 & 0\\0 & 2\end{bmatrix} - \begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}0 & 0\\0 & -3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies y=0 \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\0\end{bmatrix}$$

where x is a free variable.

Any vector in span$\{\begin{bmatrix}1\\0\end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 5$.

$\lambda = 2$:

$$\begin{bmatrix}3 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies x=0 \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\y\end{bmatrix}$$

where y is a free variable.

Any vector in span$\{\begin{bmatrix}0\\1\end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 2$.

(b) $\mathbf{A} = \begin{bmatrix}22 & 6\\6 & 13\end{bmatrix}$

Solution:

Here, it is hard to guess the answers.

$$\det\left(\begin{bmatrix}22-\lambda & 6\\6 & 13-\lambda\end{bmatrix}\right) = 0$$

$$\begin{aligned}(22-\lambda)(13-\lambda)-36 &= 0\\ 250-35\lambda+\lambda^2 &= 0\\ (\lambda-10)(\lambda-25) &= 0\\ \implies \lambda &= 10, 25\end{aligned}$$

$\lambda = 10$:

$$\mathbf{A}\vec{x} = 10\vec{x} \implies (\mathbf{A}-10\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix}22 & 6\\6 & 13\end{bmatrix} - \begin{bmatrix}10 & 0\\0 & 10\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}12 & 6\\6 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\implies \begin{bmatrix}2 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies 2x+y=0 \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\-2x\end{bmatrix}$$

where x is a free variable.

Any vector that lies in span$\{\begin{bmatrix}1\\-2\end{bmatrix}\}$ is an eigenvector with corresponding eigenvalue $\lambda = 10$.

$\lambda = 25$:

$$\mathbf{A}\vec{x} = 25\vec{x} \implies (\mathbf{A}-25\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix}22 & 6\\6 & 13\end{bmatrix} - \begin{bmatrix}25 & 0\\0 & 25\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}-3 & 6\\6 & -12\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\implies \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2y = x \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix}$$

where y is a free variable.

Any vector that lies in span$\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 25$.

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Solution:

This can also be seen by inspection. The matrix is not invertible since the first two rows are linearly dependent. Therefore, there must be a 0 eigenvalue. This has the eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, which belongs in the **nullspace of the matrix**.

The other eigenvector can be seen by noticing that the second row is twice the first. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a good guess to try and indeed it works with $\lambda = 5$.

Alternatively, we can explicitly calculate.

$$\det(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}) = 0$$

$$\begin{aligned} (1-\lambda)(4-\lambda) - 4 = 0 \\ \lambda^2 - 5\lambda = 0 \implies \lambda(\lambda - 5) = 0 \\ \lambda = 0, 5 \end{aligned}$$

$\lambda = 0$:

$$\mathbf{A}\vec{x} = 0\vec{x} \implies \mathbf{A}\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = -2y \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix}$$

where y is a free variable.

Any vector that lies in span$\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 0$.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies y = 2x \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$$

where x is a free variable.

Any vector that lies in span$\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 5$.

(d) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

Solution:

Recall the definition of a matrix subspace from Note 8. A subspace $\mathbb{U}$ consists of a subset of the vector space $\mathbb{V}$ if it contains the zero vector, is closed under scalar multiplication, and is closed under vector addition.

i. The zero vector is contained in this set since $\mathbf{A}\vec{0} = \vec{0} = \lambda\vec{0}$.

ii. Let $\vec{v}_1$ and $\vec{v}_2$ be members of the set. Let $\vec{u} = \alpha\vec{v}_1$. Now, $\mathbf{A}\vec{u} = \mathbf{A}\alpha\vec{v}_1 = \alpha\mathbf{A}\vec{v}_1 = \alpha\lambda\vec{v}_1 = \lambda\vec{u}$. Hence, $\vec{u}$ is a member of the set as well and the set is closed under scalar multiplication.

iii. Observe below that the set is closed under vector addition as well.

$$\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2 = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2)$$

Hence, the set defined in the question satisfies the properties of a subspace and is consequently a subspace of $\mathbb{R}^n$.

3. Can You Hear the Shape of a Drum?

This problem is inspired by a popular problem posed by Mark Kac in his article "Can you hear the shape of a drum?"[1] Kac's question was about different shapes of drums. Here's what he wanted to know: if the shape of a drum defines the sound that's made when we strike it, can we listen to the drum and automatically infer its shape? Deep down, this is really a question about eigenvalues and eigenvectors of a matrix. The vibrational dynamics of a particularly shaped drum membrane can be captured by a system of linear equations represented by a matrix. The eigenvalues and eigenvectors of this matrix reveal interesting properties about the drum that will help us answer the question: can we hear its shape?

Before we answer this question, we will first consider a simpler problem of modeling the vibration of a one dimensional violin string.

We'll use a model of vibration in one dimension given by the equation,

$$\frac{d^2u(x)}{dx^2} + \lambda u(x) = 0 \tag{1}$$

Here, u is the amount of vertical displacement of the string at a particular location x, and λ is an unknown parameter (which will turn out to be an eigenvalue, as you will see). We can make the approximation:

$$\frac{d^2u}{dx^2} \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \tag{2}$$

where h is some small constant. This approximation just follows from the limit based definition of the derivative, and allows us to discretize a continuous problem. You can think of h as the distance between two points in the discretized model of the string.

[1] Marc Kac, Can one hear the shape of a drum?, Amer. Math. Monthly 73 (1966), 1-23.

(a) First, look at the diagram below that shows a 1D violin string. To analyze the problem, we will consider the vibration of the string at 3 points on the string (in orange), while the 2 end points remain fixed (in purple). Assume that the length of the string is 1 meter (even though that's kind of long for a violin...). Furthermore, we will assume that all 5 points are equally spaced ($\frac{1}{4}$ meters apart). There are therefore 3 unknowns: $u[1]$, $u[2]$, and $u[3]$ (note: 1, 2, 3 stand for the labels given to the points and not their respective distances from point 0). Use equations 1 and 2 with $h = \frac{1}{4}$ to derive a matrix vector equation that describes the vibration of the string at the 3 points.

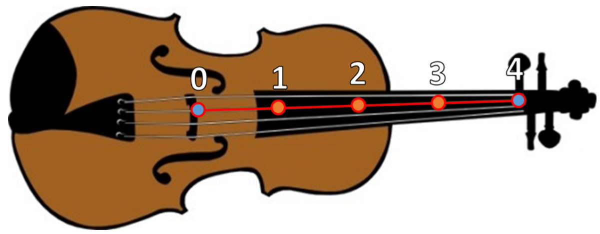


Figure 1: A 5-point model of a violin string.

Solution:

$$-\lambda \begin{bmatrix} u[1] \\ u[2] \\ u[3] \end{bmatrix} = \begin{bmatrix} -32 & 16 & 0 \\ 16 & -32 & 16 \\ 0 & 16 & -32 \end{bmatrix} \begin{bmatrix} u[1] \\ u[2] \\ u[3] \end{bmatrix}$$

(b) The 3 eigenvalues of the matrix in part a) happen to be

$$\lambda_1 = \frac{\sqrt{2}-2}{0.25^2} = -9.37...$$

$$\lambda_2 = \frac{-2}{0.25^2} = -32$$

$$\lambda_3 = \frac{-\sqrt{2}-2}{0.25^2} = -54.6...$$

For the vibrating string, find the 3 corresponding eigenvectors. What do these vectors look like?

Solution:

$$\vec{u}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\vec{u}_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}$$

Should be sinusoidal, with zeros at boundaries.

(c) What do you think the eigenvalues mean for our vibrating string? (Hint: what does a larger eigenvalue seem to indicate about the corresponding eigenvector?)

Solution:

As the eigenvalue gets larger, the string should be more "wiggly." This corresponds to a vibration at a higher frequency (also called pitch). Out of the scope of this class, the eigenvalue also corresponds to the "energy" in the mode. You would need information about the density of the string and an air damping constant to derive the sound (or pitch) generated by the vibration of the string.

In two dimensions, we can model vibration with the following equation instead:

$$\nabla^2 u(x,y) + \lambda u(x,y) = 0$$

The "∇^2" is an operator called the "Laplacian," and just stands for taking the 2nd x-partial-derivative and adding it to the 2nd y-partial-derivative. We can similarly approximate the laplacian operator with the following discretized difference equation:

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \frac{u(x+h,y) + u(x,y+h) - 4u(x,y) + u(x,y-h) + u(x-h,y)}{h^2}$$

Using what you know from part (a) of this problem, we will write down the 5-point finite difference equation for a 5×5 square drum in the form of a matrix problem so that it has the same form as

$$-\lambda \vec{u} = \mathbf{A}\vec{u}$$

In this formulation, as in the 1D formulation, each row of **A** will correspond to the equation of motion for one point on the model. In our 5×5 grid, we will be modeling the motion of the inner 3×3 grid, since we will assume the membrane is fixed on the outer border. Since there are 9 points that we are modeling, this corresponds to 9 equations and 9 unknowns, so **A** should be 9×9.

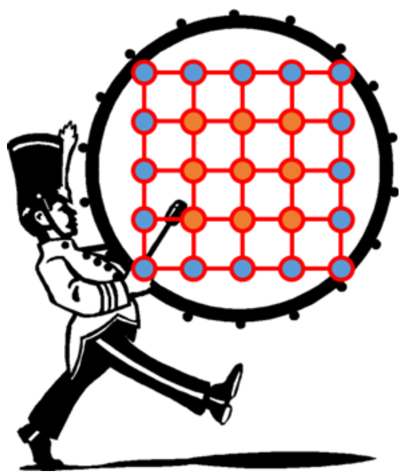


Figure 2: A 25-point model of a drum membrane.

(d) Based on our intuition from the 1D problem, what do the eigenvalues and eigenvectors correspond to in the 2D problem?

Solution: Eigenvectors represent a 2D function which is a possible standing wave for the particular shape of drum that we have. Eigenvalues correspond to the frequency-squared of the corresponding eigenvector.

(e) Write down the 9×9 matrix, $\mathbf{A}$, for the drum in Figure 2. It should have some symmetry, but be careful with the diagonals.

Solution:

$$\mathbf{A} = \frac{1}{h^2}\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}$$

Note the missing zeros on the $+1$ and -1 diagonals. Since the side length of the square mesh is not given, the spacing, h, could be chosen arbitrarily by the student. If $h = 0.25$ is used (as in previous parts), the -4's will change to -64's, and 1's will change to 16's. Eigenvalues will scale with $\frac{1}{h^2}$ as well.

(f) In the IPython Notebook, implement a function to solve the finite difference problem for a square drum of any side-length (though keep the side-length short at first, so that you don't run into memory problems!). What are the eigenvalues of the 5×5 drum?

Solution:

See the IPython notebook.

(g) Using some of the built-in functionality in the notebook, you can construct a drum with any polygonal shape. There are two shapes already implemented, with the shapes shown below. The code already included will construct the $\mathbf{A}$ matrix given a polygon and a grid. Find the first 10 vibrational modes of each drum, and the associated eigenvalues (this is analogous to finding the first 10 eigenvectors of each $\mathbf{A}$ matrix, and the associated eigenvalues). Plot the 0th, 4th, and 8th modes using a contour plot.

Solution: See the IPython notebook.

(h) These two drums are different shapes. Do they sound the same? Why or why not? Can you hear the shape of a drum?

Solution:

The drums given in the notebook sound the same. We know this because the eigenvalues of each drum are identical (to numerical precision implemented by your Python installation). Therefore, you cannot hear the shape of a drum!

4. The Dynamics of Romeo and Juliet's Love Affair

Learning Goal: *Eigenvalues and eigenvectors of state transition matrices tend to reveal useful information about the dynamical systems they model. This problem serves as an example of extracting useful information through analysis of the eigenvalues of the state transition matrix of a dynamical system.*

In this problem, we will study a discrete-time model of the dynamics of Romeo and Juliet's love affair—adapted from Steven H. Strogatz's original paper, *Love Affairs and Differential Equations*, Mathematics Magazine, 61(1), p.35, 1988, which describes a continuous-time model.

Let $R[n]$ denote Romeo's feelings about Juliet on day n, and let $J[n]$ denote Juliet's feelings about Romeo on day n, where $R[n]$ and $J[n]$ are **scalars**. The **sign** of $R[n]$ (or $J[n]$) indicates like or dislike. For example, if $R[n] > 0$, it means Romeo likes Juliet. On the other hand, $R[n] < 0$ indicates that Romeo dislikes Juliet. $R[n] = 0$ indicates that Romeo has a neutral stance towards Juliet.

The **magnitude** (i.e. absolute value) of $R[n]$ (or $J[n]$) represents the intensity of that feeling. For example, a larger magnitude of $R[n]$ means that Romeo has a stronger emotion towards Juliet (strong love if $R[n] > 0$ or strong hatred if $R[n] < 0$). Similar interpretations hold for $J[n]$.

We model the dynamics of Romeo and Juliet's relationship using the following linear system:

$$R[n+1] = a\,R[n] + b\,J[n], \quad n = 0,1,2,\ldots$$

and

$$J[n+1] = c\,R[n] + d\,J[n], \quad n = 0,1,2,\ldots,$$

which we can rewrite as

$$\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n],$$

where $\vec{s}[n] = \begin{bmatrix} R[n] \\ J[n] \end{bmatrix}$ denotes the state vector and $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denotes the state transition matrix for our dynamic system model.

The selection of the parameters a,b,c,d results in different dynamic scenarios. The fate of Romeo and Juliet's relationship depends on these model parameters (i.e. a,b,c,d) in the state transition matrix and the initial state ($\vec{s}[0]$). In this problem, we'll explore some of these possibilities.

(a) Consider the case where $a+b = c+d$ in the state-transition matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_1.
Show that

$$\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_2.
Now, express the first and second eigenvalues and their eigenspaces in terms of the parameters $a,b,c,$ and d.
Hint: *Consider $\mathbf{A}\vec{v}_1$. Is it equal to a scalar multiple of $\vec{v}_1$? Repeat a similar process for $\vec{v}_2$.*

Solution:

$$\begin{aligned} \mathbf{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} a+b \\ c+d \end{bmatrix} \\ &= (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (c+d)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Let $\lambda_1 = a+b = c+d$. So you find that $\begin{bmatrix}1 & 1\end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_1. So, we find that,

$$\left(\lambda_1 = a+b = c+d,\ Eigenspace(\lambda_1) = span\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\right)$$

To determine the other eigenvalues and corresponding eigenvectors $(\lambda_2, \vec{v}_2)$, we test the assumption that $\vec{v}_2 = \begin{bmatrix}b\\-c\end{bmatrix}$. Note that by modifying the constraint $a+b = c+d$, we can also get $a-c = d-b$, which helps simplify the following:

$$\begin{aligned}\mathbf{A}\begin{bmatrix}b\\-c\end{bmatrix} &= \begin{bmatrix}ab-bc\\cb-dc\end{bmatrix}\\ &= \begin{bmatrix}b(a-c)\\-c(d-b)\end{bmatrix}\\ &= (a-c)\begin{bmatrix}b\\-c\end{bmatrix}\\ &= (d-b)\begin{bmatrix}b\\-c\end{bmatrix}\end{aligned}$$

Therefore, we have our second eigenvalue and corresponding eigensapce:

$$\left(\lambda_2 = a-c = d-b,\ Eigenspace(\lambda_2) = span\left(\begin{bmatrix}b\\-c\end{bmatrix}\right)\right).$$

For parts (b) - (e), consider the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix}0.75 & 0.25\\0.25 & 0.75\end{bmatrix}$$

(b) Determine the eigenvalues and corresponding eigenvectors (i.e. $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

Solution:

From the results of part (a), we know that the eigenvalues and eigenvectors of this matrix are

$$\left(\lambda_1 = a+b = 0.75+0.25 = 1, \vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}\right)$$

and

$$\left(\lambda_2 = a-c = 0.75-0.25 = 0.5, \vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}\right).$$

Note: If your choice of eigenvector $\vec{v}_1$ and $\vec{v}_2$ is a scaled version of the ones given in this solution, that is fine.

(c) Determine all of the non-zero *steady states* of the system. That is, find all possible state vectors $\vec{s}_*$ such that if Romeo and Juliet start at, or enter, any of those state vectors, their states will stay in place forever: $\{\vec{s}_* \mid \mathbf{A}\vec{s}_* = \vec{s}_*\}$.

Solution: Any $\vec{s}_* \in \text{span}\{\vec{v}_1\}$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, is the eigenvector which corresponds to the steady state, because $\vec{v}_1$ corresponds to the eigenvalue $\lambda_1 = 1$.

(d) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution:

We note that $\vec{s}[0] \in \text{span}\{\vec{v}_2\}$. Therefore,

$$\begin{aligned} \vec{s}[1] &= \mathbf{A}\vec{s}[0] \\ &= \alpha\lambda_2\vec{v}_2 \end{aligned}$$

where α is the scalar that expresses $\vec{s}[0]$ as a scaled version of $\vec{v}_2$.

If we continue to apply the state transition matrix, we will see that for this $\vec{s}[0]$,

$$\begin{aligned} \vec{s}[n] &= \mathbf{A}^n\vec{s}[0] \\ &= \alpha\lambda_2^n\vec{v}_2 \end{aligned}$$

In this case $\lambda_2 = 0.5$. This means that as $n \to \infty$, $\lambda_2^n \to 0$.

Therefore,

$$\begin{aligned} \vec{s}[n] &= \alpha\lambda_2^n\vec{v}_2 \\ &= \alpha \cdot 0 \cdot \vec{v}_2 \\ &= \vec{0} \end{aligned}$$

which means that

$$\lim_{n\to\infty}(R[n], J[n]) = (0,0)$$

So, ultimately, Romeo and Juliet will become neutral to each other.

(e) Suppose the initial state is $\vec{s}[0] = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Hint: *Can you use what you learned about the eigenvectors of* $\mathbf{A}$ *(in parts c and d) to help you solve this problem? You can represent the starting state as a linear combination of eigenvectors* $\vec{v}_1$ *and* $\vec{v}_2$*.*

Solution:

We must express the initial state vector as a linear combination of the eigenvectors. That is, we must solve the system of linear equations

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \vec{\alpha} = \vec{s}[0]$$
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

You can row-reduce to find the solution:

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

Therefore, the starting state vector is given by

$$\vec{s}[0] = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = 4\vec{v}_1 - 1\vec{v}_2$$

If we apply the state transition matrix,

$$\begin{aligned} \vec{s}[1] &= \mathbf{A}\vec{s}[0] \\ &= \alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2 \\ &= 4\lambda_1 \vec{v}_1 - 1\lambda_2 \vec{v}_2 \end{aligned}$$

If we continue to apply the state transition matrix, we find:

$$\begin{aligned} \vec{s}[n] &= \alpha_1 \lambda_1^n \vec{v}_1 + \alpha_2 \lambda_2^n \vec{v}_2 \\ &= 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \left(\frac{1}{2}\right)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 - \left(\frac{1}{2}\right)^n \\ 4 + \left(\frac{1}{2}\right)^n \end{bmatrix} \end{aligned}$$

Taking the limit as $n \to \infty$,

$$\lim_{n\to\infty} \vec{s}[n] = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

Romeo and Juliet's relationship converges to the state vector $\vec{s}[n] = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

Now suppose we have the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Use this state-transition matrix for parts (f) - (h).

(f) Determine the eigenvalues and corresponding eigenvectors (i.e. $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$) for this system. Note that this matrix is **a special case** of the matrix explored in part (a), so you can use results from that part to help you.

Solution: From the results of part (a), we know that the eigenvalues and corresponding eigenvectors of this matrix are

$$\left(\lambda_1 = a + b = 1 + 1 = 2, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

$$\left(\lambda_2 = a - c = 1 - 1 = 0, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

(g) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution: The initial state $\vec{s}[0]$ lies in the span of the eigenvector $\vec{v}_2$, which has eigenvalue $\lambda_2 = 0$. Thus, $\vec{s}[1] = \mathbf{A}\vec{s}[0]\vec{0}$. The state will remain at $\vec{0}$ for all subsequent time steps, i.e.

$$\vec{s}[n] = \vec{0}, n \geq 1$$

Therefore, Romeo and Juliet become neutral towards each other in the long run, i.e.

$$\lim_{n\to\infty}(R[n], J[n]) = (0,0)$$

(h) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution: We note that $\vec{s}[0] \in \text{span}\{\vec{v}_1\}$. Therefore,

$$\begin{aligned}\vec{s}[1] &= \mathbf{A}\vec{s}[0] \\ &= \alpha\lambda_1\vec{v}_1\end{aligned}$$

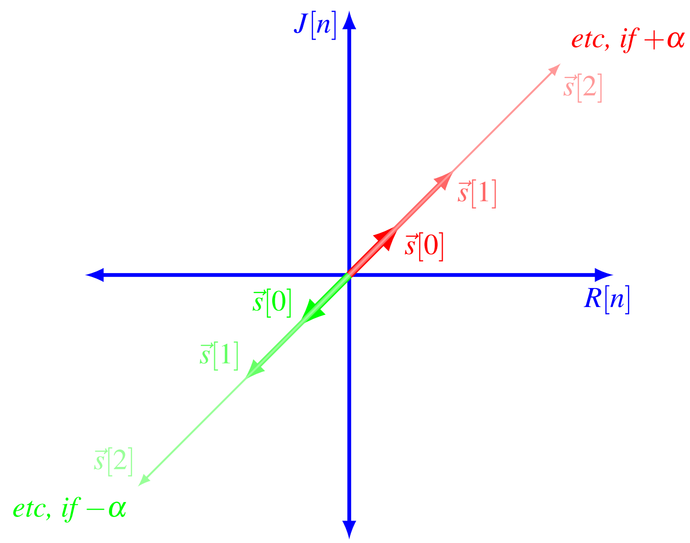
where α is the scalar that expresses $\vec{s}[0]$ as a scaled version of $\vec{v}_1$.
If we continue to apply the state transition matrix, we will see that for this $\vec{s}[0]$,

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n\vec{s}[0] \\ &= \alpha\lambda_1^n\vec{v}_1\end{aligned}$$

In this problem, $\lambda_1 = 2$. Therefore,

$$\vec{s}[n] = \alpha 2^n \vec{v}_1$$

This means that as $n \to \infty$, $\lambda_1^n \to \infty$. Essentially, the elements of the state vector continue to double at each time step and grow without bound to either $+\infty$ or $-\infty$.
Therefore, what happens to Romeo and Juliet depends on $\vec{s}[0]$. If $\vec{s}[0]$ is in the first quadrant, Romeo and Juliet will become "infinitely" in love with each other. On the other hand, if $\vec{s}[0]$ is in the third quadrant, then Romeo and Juliet will have "infinite" hatred for each other. Graphically, the dynamics of Romeo and Juliet's love affair for this example are illustrated below. The red vectors are the first three state vectors corresponding to the case where α is a *positive* value and therefore $\vec{s}[0]$ is in the first quadrant. Similarly, the green vectors are the first three state vectors corresponding to the case where α is a *negative* value and therefore $\vec{s}[0]$ is in the third quadrant.



Finally, we consider the case where we have the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

Use this state-transition matrix for parts (i) - (k).

(i) Determine the eigenvalues and corresponding eigenvectors (i.e. $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$) for this system. Note that this matrix is **a special case** of the matrix explored in part (a), so you can use results from that part to help you.

Solution: From the results of part (a), we know that the eigenvalues and corresponding eigenvectors of this matrix are

$$\left(\lambda_1 = a + b = 1 - 2 = -1, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

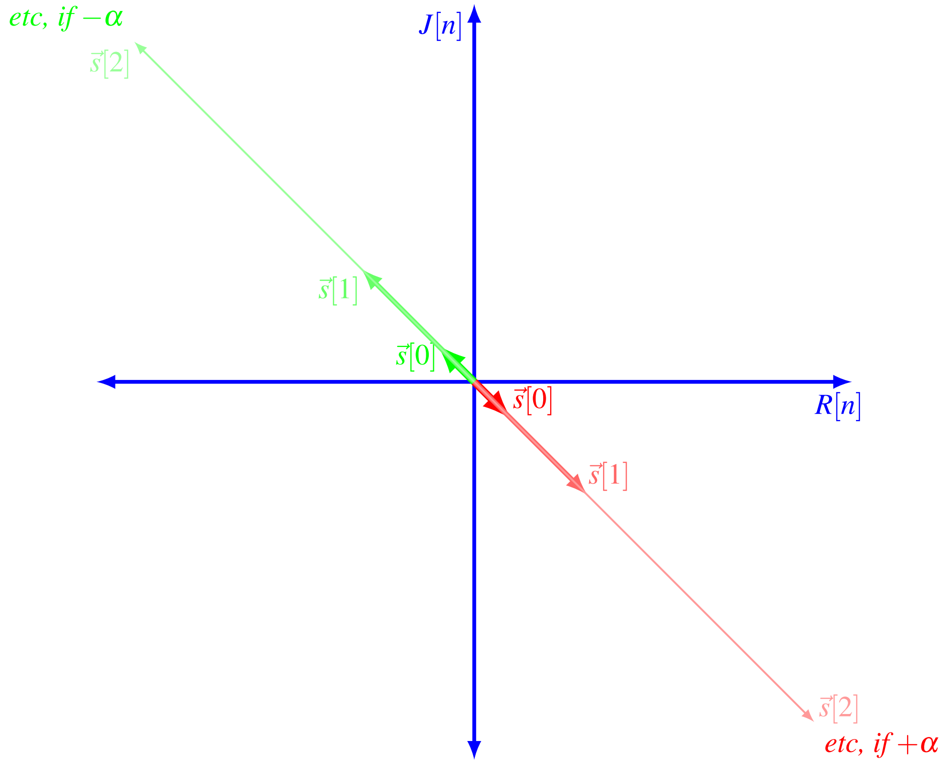
$$\left(\lambda_2 = a - c = 1 - (-2) = 3, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

(j) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] = \begin{bmatrix} R[0] \\ J[0] \end{bmatrix}$, where $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time if $R[0] > 0$ and $J[0] < 0$? What about if $R[0] < 0$ and $J[0] > 0$? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution: The initial state $\vec{s}[0]$ lies in the span of the eigenvector $\vec{v}_2$, which has eigenvalue $\lambda_2 = 3$. Using similar methods to the solutions in part (d) and part (h), we can see that (for a given scalar α):

$$\begin{aligned} \vec{s}[n] &= \mathbf{A}^n \vec{s}[0] \\ &= \alpha \lambda_2^n \vec{v}_2 \\ &= \alpha 3^n \vec{v}_2 \end{aligned}$$

There are two cases of long-term behavior.



Suppose, initially, that $R[0] > 0$ and $J[0] < 0$ (corresponding to $\alpha > 0$). Then as $n \to \infty$, $R[n] \to \infty$ and $J[n] \to -\infty$. Romeo will have "infinite" love for Juliet, while Juliet will have "infinite" hatred for Romeo. This case is illustrated with red state vectors in the figure above.

Conversely, if initially $R[0] < 0$ and $J[0] > 0$ (corresponding to $\alpha < 0$), then as $n \to \infty$, $R[n] \to -\infty$ and $J[n] \to \infty$. Now Romeo would have "infinite" hatred for Juliet, while Juliet would have "infinite" love for Romeo. This case is illustrated with green state vectors in the figure above.

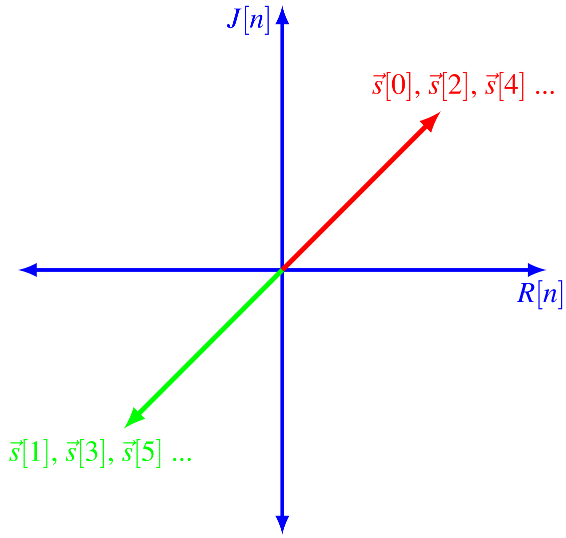
(k) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution: The initial state $\vec{s}[0]$ lies in the span of the eigenvector $\vec{v}_1$, which has eigenvalue $\lambda_1 = -1$. As with parts (d), (g), and (i), we can see that (for a given scalar α):

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n \vec{s}[0] \\ &= \alpha \lambda_1^n \vec{v}_1 \\ &= \alpha(-1)^n \vec{v}_1\end{aligned}$$

The elements of the state vector continue to switch signs at each time step, while keeping the same magnitude.

Essentially, Romeo and Juliet maintain the same intensity (i.e. absolute value or magnitude) of feeling, but they keep changing their mind about whether that feeling is like or dislike at each time step. Note that $R[0]$ and $J[0]$ have the same sign, so they both either like each other or dislike each other at a given time step n. In the figure below for an arbitrary positive valued α, we see that the state vector flip-flops between the red vector in the first quadrant for even n and the green vector in the third quadrant for odd n.



5. Traffic Flows

Learning Objective: *The learning objective of this problem is to see how the concept of nullspaces can be applied to flow problems.*

Your goal is to measure the flow rates of vehicles along roads in a town. It is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by measuring the flow along only some roads. In this problem, we will explore this concept.

(a) Let's begin with a network with three intersections, A, B and C. Define the flow t_1 as the rate of cars (cars/hour) on the road between B and A, flow t_2 as the rate on the road between C and B, and flow t_3 as the rate on the road between C and A.

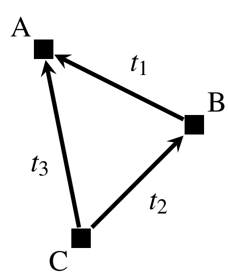


Figure 3: A simple road network.

(Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from A to C, then $t_3 = -100$. The flows now are not fractions of water in reservoirs as in the pumps setting, but numbers of cars.)

We assume the "flow conservation" constraints: the net number of cars per hour flowing into each intersection is zero. For example at intersection B, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3). If we can, find the values of t_2 and t_3.

Solution:

Yes, since we know that $t_1 = t_2 = -t_3$, so we must have $t_2 = 10$ and $t_3 = -10$.

(b) Now suppose we have a larger network, as shown in Figure 4.

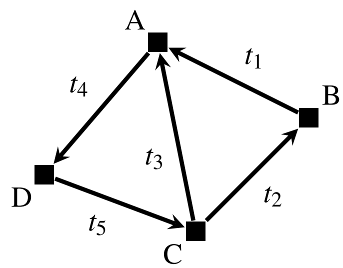


Figure 4: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads CA (measuring t_3) and DC (measuring t_5). A Stanford student claims that we need two sensors placed on the roads CB (measuring t_2) and BA (measuring t_1). Write out the system of linear equations that represents this flow graph. Is it possible to determine all traffic flows, $\left[t_1, t_2, t_3, t_4, t_5\right]^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion? *Hint: This can be solved just writing out the relevant equations and reasoning about them.*

Solution: Since we have 4 intersections, we can write 4 linear equations describing the flows into and out of each intersection. We know that the flows into and out of an intersection must sum to 0. The set of linear equations that represents this flow graph is:

$$\begin{cases} t_1 + t_3 - t_4 = 0 \\ t_2 - t_1 = 0 \\ t_5 - t_2 - t_3 = 0 \\ t_4 - t_5 = 0 \end{cases}$$

The Stanford student is wrong. Observing t_1 and t_2 is not sufficient, as t_3, t_4 and t_5 can still not be uniquely determined. Specifically, for any $\alpha \in \mathbb{R}$, the following flow satisfies the constraints and the measurements:

$$t_4 = \alpha$$
$$t_5 = \alpha$$
$$t_3 = \alpha - t_1$$

On the other hand, if we're given t_3 and t_5, we can uniquely solve for all the traffic densities as follows since we know the flow conservation constraints. From the set of linear equations we obtain:

$$\begin{aligned} t_1 &= t_5 - t_3 \\ t_2 &= t_5 - t_3 \\ t_4 &= t_5 \end{aligned}$$

This is related to the fact that t_3 and t_5 are parts of different loops in the graph, whereas t_1 and t_2 are in the same loop, so measuring both of them would not give additional information.

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) i.e. the system of equations from part (b) as a matrix equation.

Construct a 4×5 matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$:

$$\begin{bmatrix} & & \mathbf{B} & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

represents the flow conservation constraints for the network in Figure 4.

*Hint: You can construct $\mathbf{B}$ using only $0, 1,$ and -1 entries. Each row represents the inflow/outflow of an intersection. This matrix is called the **incidence matrix**.*

Solution:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

$$\quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5$$

(The rows of this matrix can be in any order and your solution can differ by a factor of -1. However, the order of the elements within the row is still important and it must match the order of the elements of $\vec{t}$). Each row represents an intersection, and each column represents a road between two intersections. Each 1 on a row represents a road flowing into an intersection, and each -1 represents a road flowing out of an intersection. Each -1 in a column represents the source intersection of a road (where the arrow starts), and each 1 in a column represents the destination intersection of a road (where the arrow ends).

Each column of $\mathbf{B}$ must sum to 0. We expect each column to sum to 0 (and actually have exactly one -1 and one 1).

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of all valid traffic flows for the network of Figure 4. Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix $\mathbf{B}$. That is, we can find all valid traffic flows by computing the null space of $\mathbf{B}$. What is the dimension of the nullspace?

Solution:

We use Gaussian Elimination to find the nullspace.

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \overset{R_2 \leftarrow R_2 + R_1}{\overbrace{\Rightarrow}} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \overset{R_3 \leftarrow R_2 + R_3}{\overbrace{\Rightarrow}} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\overset{R_5 \leftarrow R_4 + R_5}{\overbrace{\Rightarrow}} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_3 \leftarrow (-1) R_3}{\overbrace{\Rightarrow}} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_2 \leftarrow R_2 + R_3}{\overbrace{\Rightarrow}} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_1 \leftarrow R_1 + R_3}{\overbrace{\Rightarrow}} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that we should let $t_3 = \alpha$ and $t_5 = \beta$, where α and β are free variables. The equations are:

$$t_1 = \beta - \alpha$$
$$t_2 = \beta - \alpha$$
$$t_3 = \alpha$$
$$t_4 = \beta$$
$$t_5 = \beta$$

The dimension of the nullspace is 2 because a minimum of 2 vectors are required to span the entire nullspace.

Note: We show here, for your reference, that the space of all possible traffic flows is a subspace. You don't need to include this proof in your solution. Suppose we have a set of valid flows $\vec{t}$. Then, for any intersection, the total flow into it is the same as the total flow out of it. If we scale $\vec{t}$ by a constant a, each t_i will also get scaled by a. The total flows into and out of the intersection would be scaled by the same amount and remain equal to each other. Thus any scaling of a valid flow is still a valid flow. Suppose now we add valid flows $\vec{f_1}$ and $\vec{f_2}$ to get $\vec{t} = \vec{f_1} + \vec{f_2}$. For any intersection I,

$$\text{total flow into } I = \text{total flow into } I \text{ from } \vec{f_1} + \text{total flow into } I \text{ from } \vec{f_2}$$
$$\text{total flow out of } I = \text{total flow out of } I \text{ from } \vec{f_1} + \text{total flow out of } I \text{ from } \vec{f_2}$$

Since the total flow into I from $\vec{f_1}$ is the same as the total flow out of I from $\vec{f_1}$ and similarly for $\vec{f_2}$, the total flow into I is the same as the total flow out of I. Therefore, the sum of any two valid flows is still a valid flow. Also, $\vec{t} = \vec{0}$ is a valid flow. Therefore the set of valid flows forms a subspace.

(e) Notice that we can represent the Berkeley student's measurement as $\mathbf{M}_B\vec{t}$, where:

$$\mathbf{M}_B\vec{t} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \vec{t} = \begin{bmatrix} t_3 \\ t_5 \end{bmatrix}$$

Write a matrix $\mathbf{M}_S$ that can be used to represent the Stanford student's measurement.

Solution:

$$\mathbf{M}_S\vec{t} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \vec{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

(f) Now let us analyze more general road networks. Say there is a road network graph G, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a k-dimensional null space, does this mean measuring the flows along ***any*** k **roads** is always sufficient to recover all of the true flows? In other words, is there ever a possibility of being unable to recover the true flows depending on which k roads you choose?

If you think measuring the flows along any k roads will always work, then prove it showing various possible scenarios. Otherwise give an example showing a scenario where it does not work (such an example is called a counter example).

Hint: Consider the Stanford student's measurement from part (b).

Solution:

No, consider the network of Figure 4. The corresponding incidence matrix has a $k = 2$ dimensional null space, as you showed in part (e). However, measuring t_1 and t_2 (as the Stanford student suggested) is not sufficient, as you showed in part (b).

(g) **[Challenge, Optional]** Assume that $\vec{u}$ and $\vec{t}$ are distinct valid flows, that is $\mathbf{B}_G\vec{u} = \mathbf{B}_G\vec{t} = \vec{0}$. Can you recover all of the network's true flows if $(\vec{u} - \vec{t})$ belongs to the nullspace of $\mathbf{M}_S$?

Clarification: A "valid" flow is one that is possible without violating the constraints on the nodes (so flow in must equal to flow out). There may be many valid flows, but only one "true" flow. The "true flow" is one of many the valid flows, which represents the actual number of cars/ hour on each road.

Solution: No. If $(\vec{u} - \vec{t})$ is in the nullspace of $\mathbf{M}_S$, it means $\mathbf{M}_S(\vec{u} - \vec{t}) = \vec{0}$. In other words, $\mathbf{M}_S\vec{u} = \mathbf{M}_S\vec{t}$. This means that two different flows will give us the same measurement, so the true flow cannot be recovered.

(h) **[Challenge, Optional]** If the incidence matrix $\mathbf{B}_G$ has a k-dimensional null space, does this mean we can **always pick a set of** k **roads** such that measuring the flows along these roads is sufficient to recover the exact flows? If this is true, explain how you would pick these k roads to guarantee that you could recover the missing information. Otherwise, give a counterexample.

Solution:

Yes.

Let $\mathbf{U}$ be a matrix whose columns form a basis of the null space of $\mathbf{B}_G$, as above. The k columns of $\mathbf{U}$ are linearly independent since they form a basis. Since there are k linearly independent columns, when we run Gaussian elimination on $\mathbf{U}$, we must get k pivots. (Recall that "pivot" is the technical term for being able to row-reduce and turn a column into something that has exactly one 1 in it. The pivot is the entry that we found and turned into that 1.)

Therefore, the row space of $\mathbf{U}$ is k dimensional since there are some k linearly independent rows in $\mathbf{U}$ — namely the ones where we found pivots. Choose to measure the roads corresponding to these rows.

This will work because:

For a given valid flow $\vec{t} = \mathbf{U}\vec{x}$, the results of measuring this flow vector are $\mathbf{U}^{(k)}\vec{x}$, where the matrix $\mathbf{U}^{(k)}$ is some k linearly independent rows of $\mathbf{U}$. By construction, the $k \times k$ matrix $\mathbf{U}^{(k)}$ has all linearly independent rows, so we can invert $\mathbf{U}^{(k)}$ to find $\vec{x}$ from $\mathbf{U}^{(k)}\vec{x}$ and then recover the flows along all the edges as $\mathbf{U}\vec{x}$.

This isn't the only set of k roads that will work. But it does provide a set of k roads that are guaranteed to work.

6. Subspaces, Bases and Dimension

For each of the sets $\mathbb{U}$ (which are subsets of $\mathbb{R}^3$) defined below, state whether $\mathbb{U}$ is a subspace of $\mathbb{R}^3$ or not. If $\mathbb{U}$ is a subspace, find a basis for it and state the dimension. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

(a) $\mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Solution: We test the three properties of a subspace:

i. Let $\vec{v}_1 = \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{u}_1 = \alpha\vec{v}_1$, where α is a scalar. Here

$$\vec{u}_1 = \alpha\vec{v}_1 = \alpha \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2(\alpha x_1+\alpha y_1) \\ \alpha x_1 \\ \alpha y_1 \end{bmatrix} = \begin{bmatrix} 2(x_u+y_u) \\ x_u \\ y_u \end{bmatrix},$$

where $x_u = \alpha x_1$ and $y_u = \alpha y_1$. Hence, $\vec{u}_1 = \alpha\vec{v}_1$ is a member of the set as well and the set is closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2(x_2+y_2) \\ x_2 \\ y_2 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 2(x_1+y_1)+2(x_2+y_2) \\ x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} 2(x_1+x_2+y_1+y_2) \\ x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} 2(x_3+y_3) \\ x_3 \\ y_3 \end{bmatrix},$$

where $x_3 = x_1 + x_2$ and $y_3 = y_1 + y_2$ Hence, $\vec{v}_3$ is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} 2(x_0+y_0) \\ x_0 \\ y_0 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$ and $y_0 = 0$. So the vector $\vec{v}_0 = \begin{bmatrix} 2(0+0) \\ 0 \\ 0 \end{bmatrix} = \vec{0}$. So the zero vector is contained in this set.

Hence we can decide that $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Any vector in the subspace can be written as:

$$\begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

where x and y are free variables. So $\mathbb{U}$ can be expressed as span $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$. Hence the basis is given by the set: $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$. Dimension = 2.

(b) **(PRACTICE/OPTIONAL)** $\mathbb{U} = \left\{\begin{bmatrix}x\\y\\z+1\end{bmatrix} \mid x,y,z \in \mathbb{R}\right\}$

Solution:

Again we check the three properties of a subspace:

i. Now let $\vec{v_1} = \begin{bmatrix}x_1\\y_1\\z_1+1\end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{u_1} = \alpha\vec{v_1}$, where α is a scalar. Here

$$\vec{u_1} = \alpha\vec{v_1} = \begin{bmatrix}\alpha x_1\\\alpha y_1\\\alpha z_1+\alpha\end{bmatrix} = \begin{bmatrix}\alpha x_1\\\alpha y_1\\(\alpha z_1+\alpha-1)+1\end{bmatrix} = \begin{bmatrix}x_u\\y_u\\z_u+1\end{bmatrix},$$

where $x_u = \alpha x_1$, $y_u = \alpha y_1$ and $z_u = \alpha z_1 + \alpha - 1$. Hence, $\vec{u_1} = \alpha\vec{v_1}$ is a member of the set as well and the set is closed under scalar multiplication.

ii. Let $\vec{v_1} = \begin{bmatrix}x_1\\y_1\\z_1+1\end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix}x_2\\y_2\\z_2+1\end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v_3} = \vec{v_1} + \vec{v_2}$:

$$\vec{v_3} = \vec{v_1} + \vec{v_2} = \begin{bmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2+2\end{bmatrix} = \begin{bmatrix}x_1+x_2\\y_1+y_2\\(z_1+z_2+1)+1\end{bmatrix} = \begin{bmatrix}x_3\\y_3\\z_3+1\end{bmatrix},$$

where $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$ and $z_3 = z_1 + z_2 + 1$. Hence, $\vec{v_3}$ is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v_0} = \begin{bmatrix}x_0\\y_0\\z_0+1\end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$, $y_0 = 0$ and $z_0 = -1$. So the vector $\vec{v_0} = \begin{bmatrix}0\\0\\-1+1\end{bmatrix} = \vec{0}$. So the zero vector is contained in this set.

Hence we can decide that $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Any vector in the subspace can be written as:

$$\begin{bmatrix}x\\y\\z+1\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + (z+1)\begin{bmatrix}0\\0\\1\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z_{new}\begin{bmatrix}0\\0\\1\end{bmatrix},$$

where x, y and $z_{new} = z+1$ are free variables. So $\mathbb{U}$ can be expressed as span $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Hence the basis is given by the set: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. The dimension is 3, which makes $\mathbb{U}$ the same as $\mathbb{R}^3$.

(c) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Solution: Again we check the three properties of a subspace:

i. Now let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1+1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{u}_1 = \alpha\vec{v}_1$, where α is a scalar. Here

$$\vec{u}_1 = \alpha\vec{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + \alpha \end{bmatrix} \neq \begin{bmatrix} x_u \\ y_u \\ x_u + 1 \end{bmatrix},$$

where $x_u = \alpha x_1$ and $y_u = \alpha y_1$. Hence, $\vec{u}_1 = \alpha\vec{v}_1$ is not a member of the set and the set is not closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1+1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ x_2+1 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1 + x_2 + 2 \end{bmatrix} \neq \begin{bmatrix} x_3 \\ y_3 \\ x_3 + 1 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, and $y_3 = y_1 + y_2$. Hence, $\vec{v}_3$ is not a member of the set and the set is not closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \\ x_0+1 \end{bmatrix}$ be a member of the set. The first and third elements cannot both be zero regardless of the value chosen for x_0. So the zero vector is not contained in this set.

Hence we can decide that $\mathbb{U}$ is not a subspace of $\mathbb{R}^3$. **Note that for full credit you only have to show that one of the properties is violated, you don't have to show all three.**

(d) **(PRACTICE, OPTIONAL)** $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+y^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Solution: Again we check the three properties of a subspace:

i. Now let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1+y_1^2 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{u}_1 = \alpha\vec{v}_1$, where α is a scalar. Here

$$\vec{u}_1 = \alpha\vec{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + \alpha y_1^2 \end{bmatrix} \neq \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + (\alpha y_1)^2 \end{bmatrix} = \begin{bmatrix} x_u \\ y_u \\ x_u + y_u^2 \end{bmatrix},$$

where $x_u = \alpha x_1$ and $y_u = \alpha y_1$. Hence, $\vec{u}_1 = \alpha\vec{v}_1$ is not a member of the set and the set is not closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1+y_1^2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ x_2+y_2^2 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1 + x_2 + y_1^2 + y_2^2 \end{bmatrix} \neq \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ (x_1 + x_2) + (y_1 + y_2)^2 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \\ x_3 + y_3^2 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, and $y_3 = y_1 + y_2$. Hence, $\vec{v}_3$ is not a member of the set and the set is not closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \\ x_0 + y_0^2 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$ and $y_0 = 0$. So the vector $\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 + 0^2 \end{bmatrix} = \vec{0}$ is contained in this set.

Since two of the three properties do not hold, we can decide that $\mathbb{U}$ is not a subspace of $\mathbb{R}^3$.

Just showing that one of the three properties is violated is enough to prove that a subset is not a subspace. However, in order to prove that a subset is a subspace, you have to show that all three properties hold.

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.