EECS 16A Designing Information Devices and Systems I
Spring 2021 Homework 6

This homework is due Friday, March 5, 2020, at 23:59.
Self-grades are due Monday, March 8, 2020, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw6.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned)

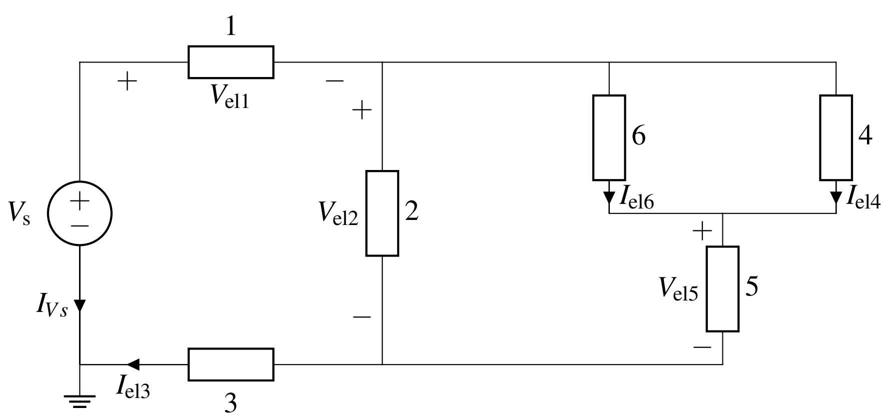
Submit the file to the appropriate assignment on Gradescope.

1. **Reading Assignment**

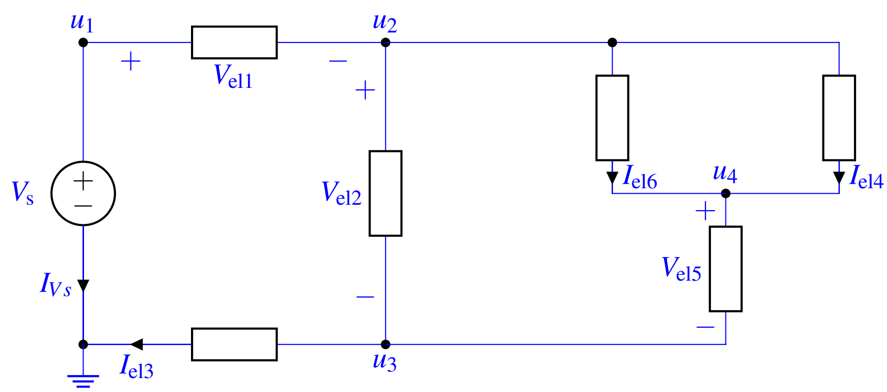
   For this homework, please read Note 11, which introduces the basics of circuit analysis and node voltage analysis. Please also read Note 12, which introduces using circuits for modelling. You are always welcome and encouraged to read beyond this as well. **Question to answer: What is the value of having a systematic procedure for solving circuits?**

2. **Intro to Circuits**

   Learning Goal: *This problem will help you practice labeling circuit elements and setting up KCL and KVL equations.*

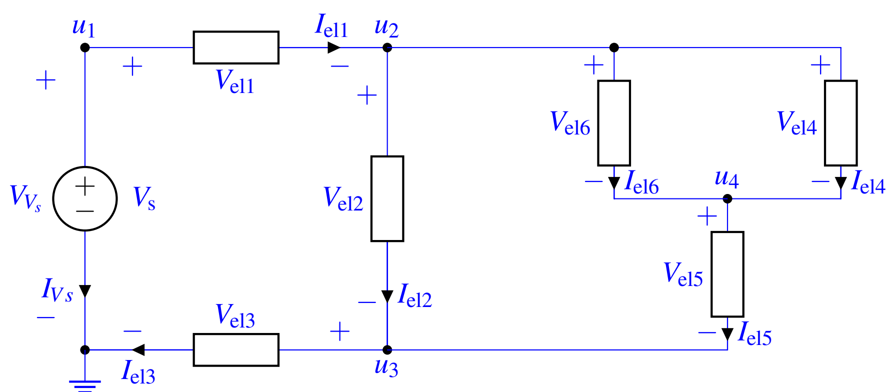


   (a) How many nodes does the above circuit have? Label them.
   Note: The ground node has been selected for you, so you don't need to label that, but you need to include it in your node count.
   Solution: There is a total of 5 nodes in the circuit, including the ground node. They are labeled u_1 - u_4 below:



(b) Notice that elements 1 - 6 and the voltage source V_s have either the *voltage across* or the *current through* them not labeled. Label the missing *voltages across* or *currents through* for elements 1 - 6, and the voltage source V_s, so that they all follow **passive sign convention**.

Solution: The passive sign convention dictates that the current flows from the positive to the negative terminal of the element (or equivalently exiting the negative terminal / entering the positive terminal if you prefer):



(c) Express all element voltages (including the element voltage across the source, V_s) as a function of node voltages. This will depend on the node labeling you chose in part (a).

Solution: For our specific node labeling we can write:

$$V_{V_s} = u_1 - 0 = u_1 (= V_s)$$

$$V_{\text{el1}} = u_1 - u_2$$

$$V_{\text{el2}} = u_2 - u_3$$

$$V_{\text{el3}} = u_3 - 0 = u_3$$

$$V_{\text{el4}} = u_2 - u_4$$

$$V_{\text{el5}} = u_4 - u_3$$

$$V_{\text{el6}} = u_2 - u_4$$

Notice that the element voltage is always of the form: $V_{\text{el}} = u_+ - u_-$.

(d) Write one KCL equation that involves the currents through elements 1 and 2.
*Hint: This will **not** be specific to your node labeling. Your answer may contain currents through other elements too.*

Solution: The only node for which we can write a KCL involving elements 1 and 2 is node u_2, since they only intersect on that node:

$$I_{el1} = I_{el2} + I_{el6} + I_{el4}$$

(e) Write a KVL equation for all the loops that contain the voltage source V_s. These equations should be a function of element voltages and the voltage source V_s.

Solution: Notice that there are in fact 3 loops that contain the voltage source V_s, for which we can write the following equations, starting each time from the ground node and ending at the ground node:

$$V_s - V_{\text{el1}} - V_{\text{el2}} - V_{\text{el3}} = 0$$
$$V_s - V_{\text{el1}} - V_{\text{el6}} - V_{\text{el5}} - V_{\text{el3}} = 0$$
$$V_s - V_{\text{el1}} - V_{\text{el4}} - V_{\text{el5}} - V_{\text{el3}} = 0$$

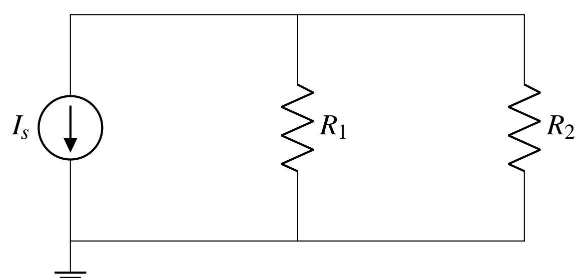
The reason this is not specific to our labeling is that the polarity of all elements is either given or set through the passive sign convention.

3. Circuit Analysis

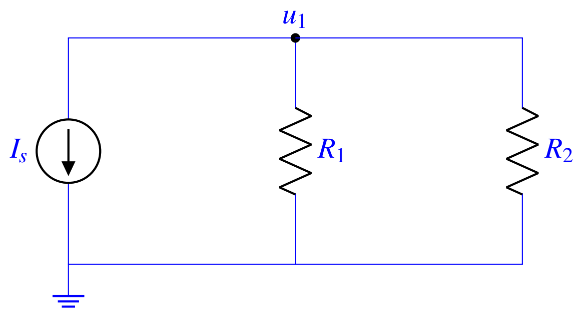
Learning Goal: *This problem will help you practice circuit analysis using NVA method.*

Using the steps outlined in lecture or in Note 11, analyze the following circuits to calculate the currents through each element and the voltages at each node. Use the ground node labelled for you. You may use a numerical tool such as IPython to solve the final system of linear equations.

(a) $I_s = 3\,\text{mA}, R_1 = 2\,\text{K}\Omega, R_2 = 4\,\text{K}\Omega$

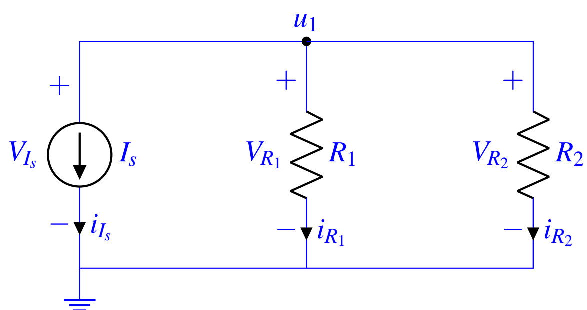


Solution:
Step 1) Define Reference Node
Select a reference (ground) node. Any node can be chosen for this purpose. This has already been done for you in this circuit.
Step 2) Label Nodes with Voltage Set by Sources
We don't have any other voltage sources in this circuit, so we can skill this step.
Step 3) Label Remaining Nodes
We only have one other node here, and we label the corresponding source u_1 (names are arbitrary, but must be unique).



Step 4) Label Element Voltages and Currents

Next we mark all element voltages and currents.
Start with the current. The direction is arbitrary (top to bottom, bottom to top, it won't matter, but stick with your choice in subsequent steps). Then mark the element voltages following the passive sign convention, i.e. the voltage and current point in the "same" direction.



Step 5) KCL Equations
Write KCL equations for all nodes with unknown voltage, which is only u_1.

$$i_{I_s} + i_1 + i_2 = 0$$

Step 6) Element Currents
Find expressions for all element currents in terms of voltage and element characteristics (e.g. Ohm's law) for all circuit elements except voltage sources. In this circuit there are three, R_1, R_2, and I_s.

$$i_{I_s} = I_s$$
$$i_{R_1} = \frac{V_{R_1}}{R_1}$$
$$i_{R_2} = \frac{V_{R_2}}{R_2}$$

Step 7) Element Voltages

Rewrite the element voltages using the node differences.

$$i_{I_s} = I_s$$
$$i_{R_1} = \frac{u_1}{R_1}$$
$$i_{R_2} = \frac{u_1}{R_2}$$

Step 8) Substitute Element Currents in KCL Equations
Now we substitute the expressions derived in Step 7 into the KCL equations from Step 5.

$$I_s + \frac{u_1}{R_1} + \frac{u_1}{R_2} = 0$$

We can isolate the unknown terms (u_1) on the left and the known on the right

$$\frac{u_1}{R_1} + \frac{u_1}{R_2} = -I_s$$

We only have one equation to solve. Setting up the matrix equation would just be the same as solving this equation. Solving for u_1, we get

$$u_1 = -I_s \frac{R_1 R_2}{R_1 + R_2}$$

Plugging in the values we were given, we get

$$u_1 = -2\,\text{mA} \frac{2\,\text{k}\Omega \cdot 4\,\text{k}\Omega}{2\,\text{k}\Omega + 4\,\text{k}\Omega}$$
$$= -4\,\text{V}$$

Node u_1 is $-4\,\text{V}$ relative to the ground node we defined. If we had defined the top node as ground, then the bottom node would have measured as $4\,\text{V}$. Similarly, as drawn, we have $V_{R_1} = V_{R_2} = -4\,\text{V}$; if we flipped the polarities, i.e. swapped + and -, we would have $V_{R_1} = V_{R_2} = 4\,\text{V}$.
The branch currents can be obtained from the node voltages and element equations. Therefore, we can write:

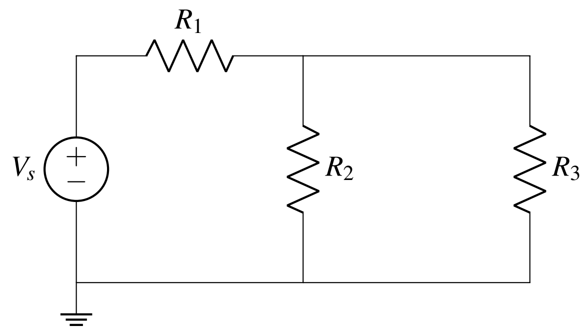
$$i_{R_1} = \frac{u_1}{R_1} = -2\,\text{mA}$$
$$i_{R_2} = \frac{u_1}{R_2} = -1\,\text{mA}$$

Note that

$$i_{R_1} = -I_s \frac{R_2}{R_1 + R_2} \qquad i_{R_2} = -I_s \frac{R_1}{R_1 + R_2}$$

These are very similar equations to the voltage divider circuit. We call this circuit a current divider.

(b) $V_s = 5\,\text{V}, R_1 = R_2 = 2\,\text{K}\Omega, R_3 = 4\,\text{K}\Omega$

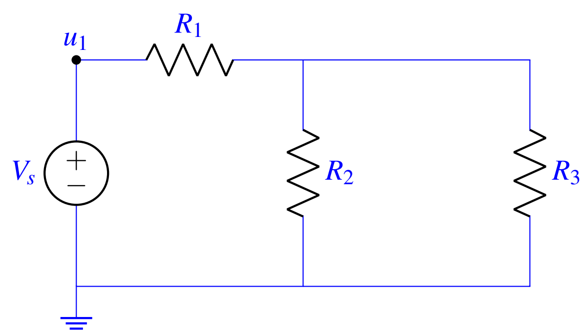


Solution:

Step 1) Define Reference Node

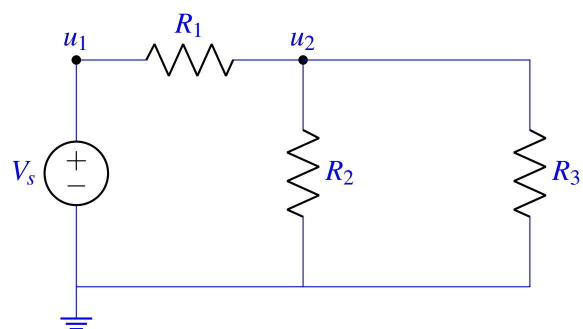
Select a reference (ground) node. Any node can be chosen for this purpose. This has already been done for you in this circuit.

Step 2) Label Nodes with Voltage Set by Sources

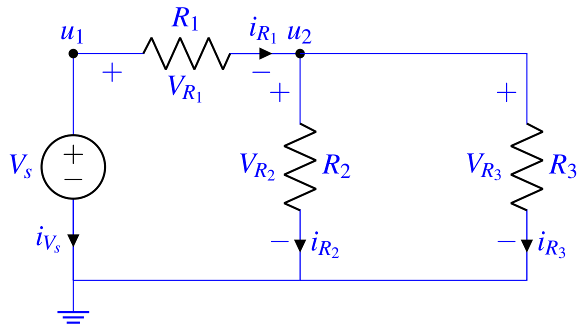


Step 3) Label Remaining Nodes

We only have one other node here, and we label the corresponding source u_2 (names are arbitrary, but must be unique).



Step 4) Label Element Voltages and Currents



Step 5) KCL Equations
Write KCL equations for all nodes with unknown voltage, which is only u_2.

$$i_1 = i_2 + i_3$$

Step 6) Element Currents
Find expressions for all element currents in terms of voltage and element characteristics (e.g. Ohm's law) for all circuit elements except voltage sources. In this circuit there are three, R_1, R_2, and R_3.

$$i_{R_1} = \frac{V_{R_1}}{R_1}$$
$$i_{R_2} = \frac{V_{R_2}}{R_2}$$
$$i_{R_3} = \frac{V_{R_3}}{R_3}$$

Step 7) Element Voltages
Rewrite the element voltages using the node differences.

$$u_1 = V_s$$
$$i_{R_1} = \frac{V_s - u_2}{R_1}$$
$$i_{R_2} = \frac{u_2}{R_2}$$
$$i_{R_3} = \frac{u_2}{R_3}$$

Step 8) Substitute Element Currents in KCL Equations
Now we substitute the expressions derived in Step 7 into the KCL equations from Step 5.

$$\frac{V_s - u_2}{R_1} = \frac{u_2}{R_2} + \frac{u_2}{R_3}$$

We can isolate the unknown terms (u_2) on the left and the known on the right

$$\frac{u_2}{R_1}+\frac{u_2}{R_2}+\frac{u_2}{R_3}=\frac{V_s}{R_1}$$

We only have one equation to solve. Setting up the matrix equation would just be the same as solving this equation. Solving for u_2, we get

$$\begin{aligned}u_2 &= \frac{V_s}{R_1}\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}} \\ &= V_s\frac{1}{1+R_1(\frac{1}{R_2}+\frac{1}{R_3})}\end{aligned}$$

Plugging in the values we were given, we get

$$\begin{aligned}u_2 &= 5\,\mathrm{V}\frac{1}{1+2\,\mathrm{k}\Omega(\frac{1}{2\,\mathrm{k}\Omega}+\frac{1}{4\,\mathrm{k}\Omega})} \\ &= 2\,\mathrm{V}\end{aligned}$$

The branch currents can be obtained from the node voltages and element equations. Therefore, we can write:

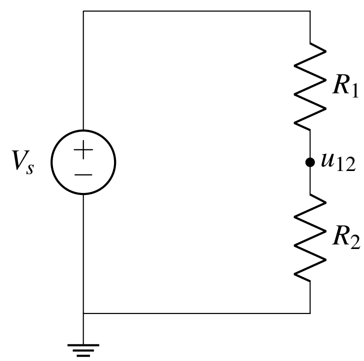
$$\begin{aligned}i_{R_1} &= \frac{u_1-u_2}{R_1}=\frac{5V-2V}{2\,\mathrm{k}\Omega}=1.5\,\mathrm{mA} \\ i_{R_2} &= \frac{u_2}{R_2}=\frac{2V}{2\,\mathrm{k}\Omega}=1\,\mathrm{mA} \\ i_{R_3} &= \frac{u_2}{R_3}=\frac{2V}{4\,\mathrm{k}\Omega}=0.5\,\mathrm{mA}\end{aligned}$$

4. Voltage divider

Learning Goal: *This problem will help you practice designing circuits under given conditions using the analysis tools you've learned.*

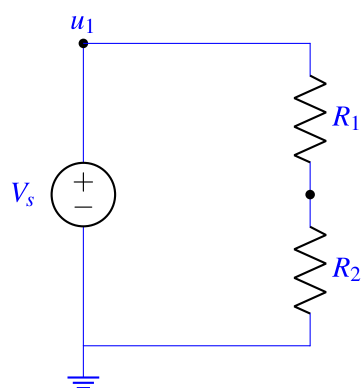
In the following parts, $V_s = 12\,\mathrm{V}$. **Choose resistance values such that the current through each element is $\leq$ 0.8A.**

(a) Select values for R_1 and R_2 in the circuit below such that $u_{12} = 6\,\mathrm{V}$. This is a **design problem**, so there can be more than one set of correct answers to this problem.

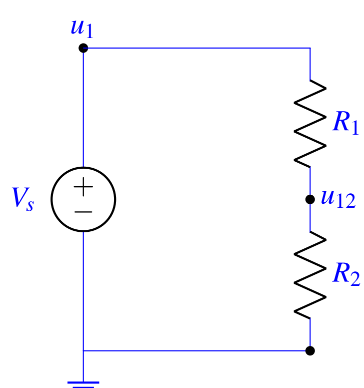


Solution: Step 1: Reference Node
We notice that the ground node has already been selected for us in the question.

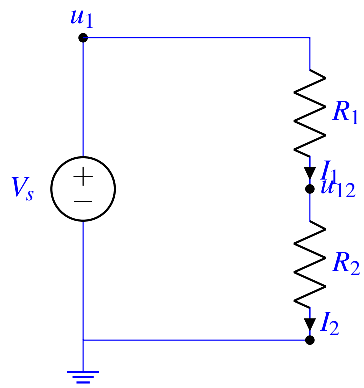
Step 2: Label nodes with voltage set by sources.



Step 3: Label other nodes.



Step 4: Label element voltages and currents.
Let V_{R1}, V_{R2} be the voltage drop across R_1 and R_2 respectively. Let I_1 be the current through R_1. Let I_2 be the current through R_2.



Step 5: KCL Equations
From KCL, at u_{12},

$$I_1 = I_2.$$

Step 6: Find element currents
Using Ohm's law, we have that

$$I_1 = \frac{V_{R1}}{R_1}$$
$$I_2 = \frac{V_{R2}}{R_2}.$$

Writing the element voltages in terms of the node voltages, we have that $V_{R1} = V_s - u_{12}$ and $V_{R2} = u_{12}$.

Step 7: Substitute element currents into KCL Equations.
Substituting back, we obtain $\frac{V_s - u_{12}}{R_1} = \frac{u_{12}}{R_2}$. Solving, we find that $u_{12} = \frac{R_2}{R_1+R_2}V_s$. Plugging in $u_{12} = 6V$ and $V_s = 12V$, we see that $R_1 = R_2$ must be true.

To choose R_1 and R_2 such that the current through each element is ≤ 0.8A, use KVL to write an expression for I_1, I_2 as a function of R_1, R_2:

$V_s - I_1R_1 - I_2R_2 = 0$, with $I_1 = I_2 = I_s$
$V_s = I_s(R_1 + R_2)$
$I_s = \frac{V_s}{(R_1+R_2)}$
We need,
$I_s \leq 0.8\text{A}$
Therefore,
$\frac{12\text{V}}{(R_1+R_2)} \leq 0.8\text{A}$
$R_1 + R_2 \geq \frac{12\text{V}}{0.8\text{A}}$
$R_1 + R_2 \geq 15\,\Omega$
As $R_1 + R_2$ must be at least $15\,\Omega$, and $R_1 = R_2$, we choose $R_1 = R_2 = 7.5\,\Omega$. Any other solution with $R_1 = R_2 = R \geq 7.5\Omega$ is also a valid solution.

5. Temperature Sensor

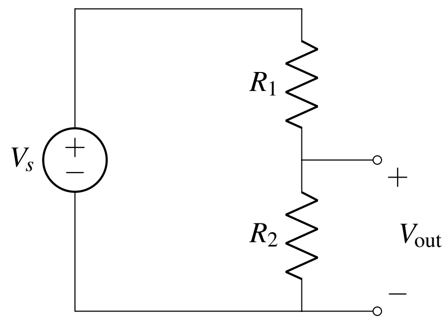
Learning Goal: *This problem will let you apply the tools we have learned so far to a real world circuits application.*

Measuring quantities in the physical world is the job of sensors. This means somehow extracting that information from the world and then converting it into a form that can be observed and processed. Electrical circuits can be very useful for doing this.

For most materials, resistance increases with increasing temperature; that is, a resistor has higher resistance when it is hot than when it is cold. This is often an annoyance to circuit designers who want their circuits to work the same way at different temperatures, but this fact can also be useful. It allows us to convert temperature, a "physical" quantity, into resistance, an "electrical" quantity, to build an electronic thermometer.

In this problem, we are going to explore how effective a particular circuit made out of various types of resistors is at allowing us to measure temperature.

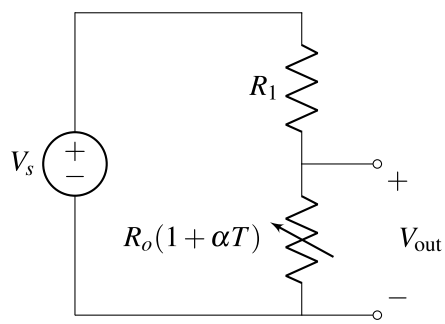
(a) Let's begin by analyzing a common topology, the voltage divider shown below. Find an equation for the voltage V_{out} in terms of R_1, R_2, and V_s.



Solution:
We recognize that this circuit is a voltage divider, we can directly write:

$$V_{\text{out}} = \frac{R_2}{R_1 + R_2} V_s$$

(b) Now let's suppose that R_1 is an ideal resistor that does not depend on temperature, but R_2 is a temperature-dependent resistor whose resistance R is set by $R = R_o(1 + \alpha T)$, where T is the absolute temperature. Find an equation for the temperature T in terms of the voltage V_{out}, V_s, R_1, R_o, and α.



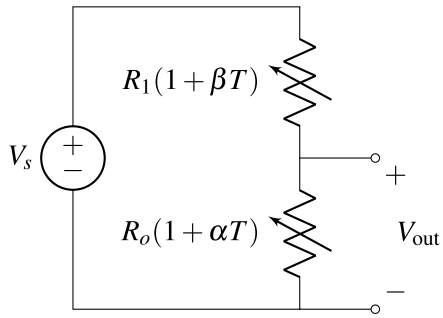
Solution:
Using the relationship from the earlier part:

$$V_{\text{out}} = \frac{R_o(1 + \alpha T)}{R_1 + R_o(1 + \alpha T)} V_s$$

$$R_1 V_{\text{out}} + R_o V_{\text{out}} + R_o \alpha T V_{\text{out}} = R_o V_s + R_o \alpha T V_s$$

$$T = \frac{(R_1 + R_o)V_{\text{out}} - R_o V_s}{R_o \alpha (V_s - V_{\text{out}})}$$

(c) It turns out that almost all resistors have some temperature dependence. Consider the same circuit as before, but now, R_1' has a temperature dependence given by $R_1' = R_1(1+\beta T)$. Find an equation for the temperature T in terms of the voltage V_{out}, R_1, R_o, V_s, α, and β.



Solution:

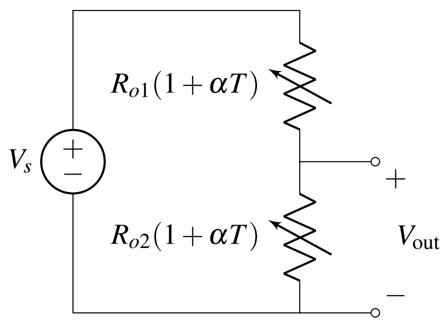
Once again using the equation for the voltage divider:

$$V_{\text{out}} = \frac{R_o(1+\alpha T)}{R_1(1+\beta T) + R_o(1+\alpha T)} V_s$$

$$R_1 V_{\text{out}} + R_1 \beta T V_{\text{out}} + R_o V_{\text{out}} + R_o \alpha T V_{\text{out}} = R_o V_s + R_o \alpha T V_s$$

$$T = \frac{(R_1 + R_o)V_{\text{out}} - R_o V_s}{R_o \alpha (V_s - V_{\text{out}}) - R_1 \beta V_{\text{out}}}$$

(d) Your colleague who has not taken EECS16A thinks that they can improve this circuit's ability to measure temperature by making both resistors depend on temperature in the same way. He hence came up with the circuit shown below, where both R_1 and R_2 have nominally different values, but both vary with temperature as a function of $(1+\alpha T)$. Can this circuit be used to measure temperature? Why or why not?



Solution: Using the equation for a voltage divider:

$$V_{\text{out}} = \frac{R_{o2}(1+\alpha T)}{R_{o1}(1+\alpha T) + R_{o2}(1+\alpha T)} V_s = \frac{R_{o2}}{R_{o1} + R_{o2}} V_s$$

Notice this circuit cannot be used to measure temperature because the output voltage is independent of temperature.

6. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.