EECS 16A Designing Information Devices and Systems I
Spring 2021 Homework 8

This homework is due Friday, March 19, 2021 at 23:59. Self-grades are due Monday, March 22, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw8.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. **Reading Assignment**

For this homework, please read Note 15: Section 15.3 to learn about superposition, a concept that can help to simplify circuit analysis. Also read Note 15: Sections 15.7-15.8, which will the explain idea of finding the equivalent resistance.

(a) As a part of superposition, you need to zero out independent sources. What circuit elements are equivalent to a zeroed voltage source and zeroed current source, respectively?

Solution:
– A zeroed voltage source is equivalent to a closed wire, since there is no voltage difference across directly connected wires.
– A zeroed current source is equivalent to an open wire, since no current can flow between disconnected leads.

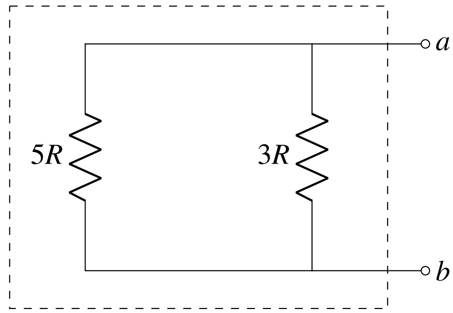
(b) If you connect three resistors (each with value R) in series, what will be the equivalent resistance? What happens if you connect these resistors in parallel?

Solution:
– If you connect three resistors R in series, their equivalent resistance is simply the summation of their values: $R_{EQ} = 3R$.
– If you connect three equal resistors R in parallel, then the current will divide equally across all three resistor paths and thus: $R_{EQ} = \frac{1}{3}R$.

2. **Equivalent Resistance**

Learning Goal: *The objective of this problem is to practice finding the equivalent to a series/parallel combination of resistors.*

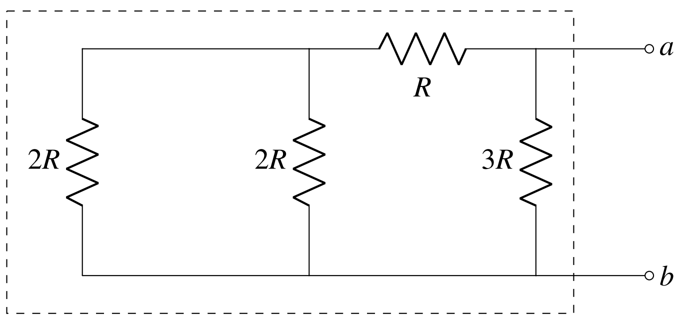
(a) Find the equivalent resistance looking in from points a and b. In other words, express the resistive network in the dashed box as one resistor.



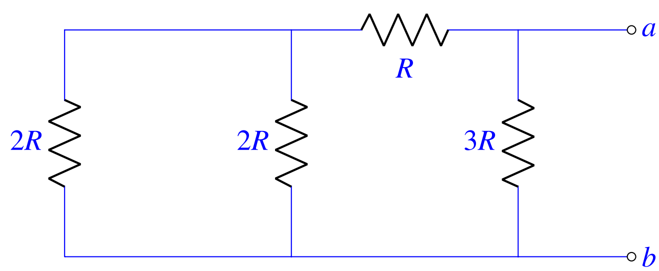
Solution:

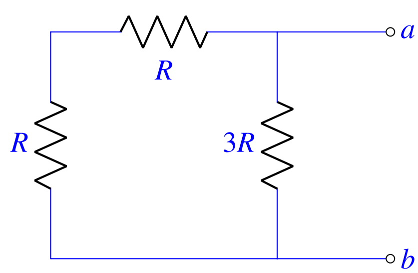
$$R_{eq} = 5R \parallel 3R = \frac{5R \cdot 3R}{5R + 3R} = \frac{15}{8}R$$

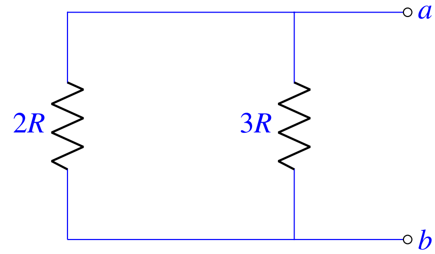
(b) Find the equivalent resistance looking in from points a and b. In other words, express the resistive network in the dashed box as one resistor.



Solution: We find the equivalent resistance for the resistors from left to right.

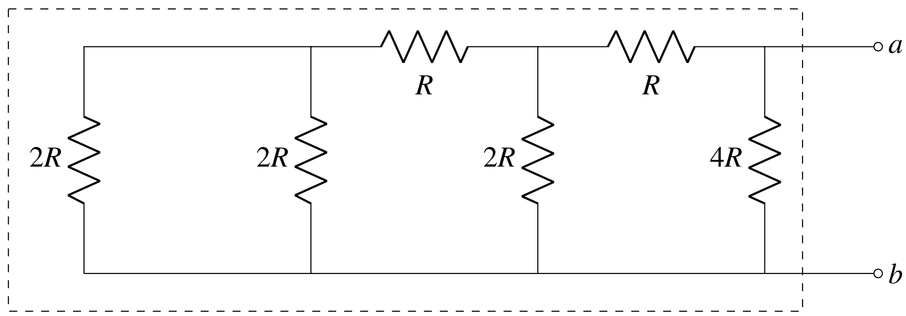




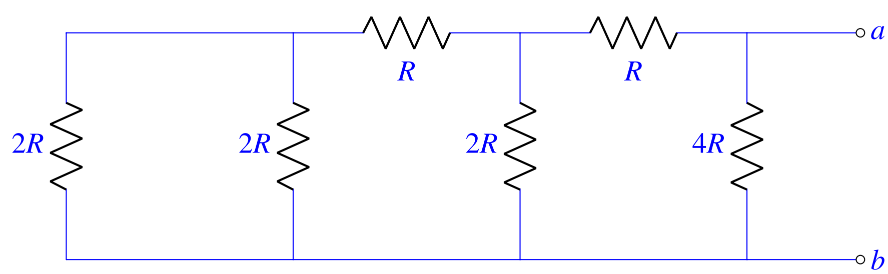


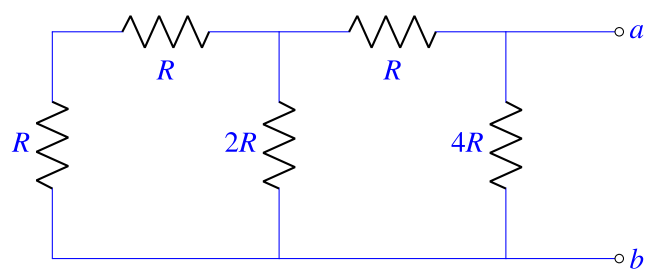
$$R_{eq} = 2R \parallel 3R = \frac{6}{5}R$$

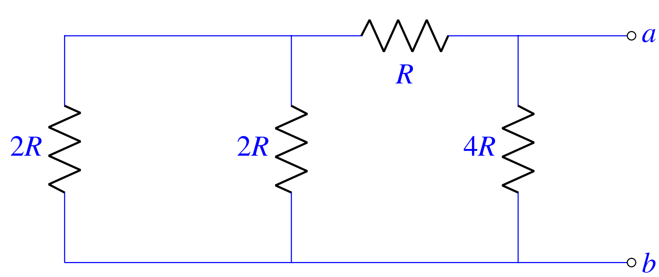
(c) Find the equivalent resistance looking in from points a and b. In other words, express the resistive network in the dashed box as one resistor.

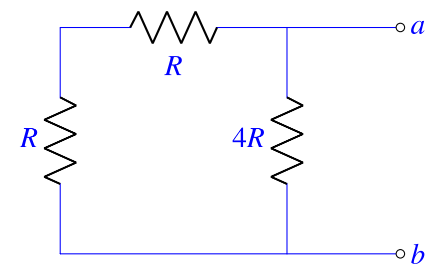


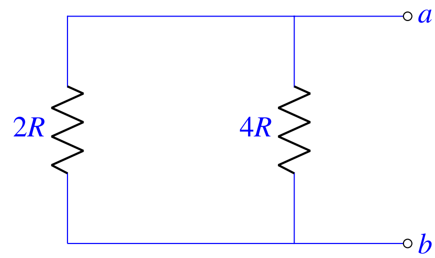
Solution: Again, we find the equivalent resistance for the resistors from left to right.





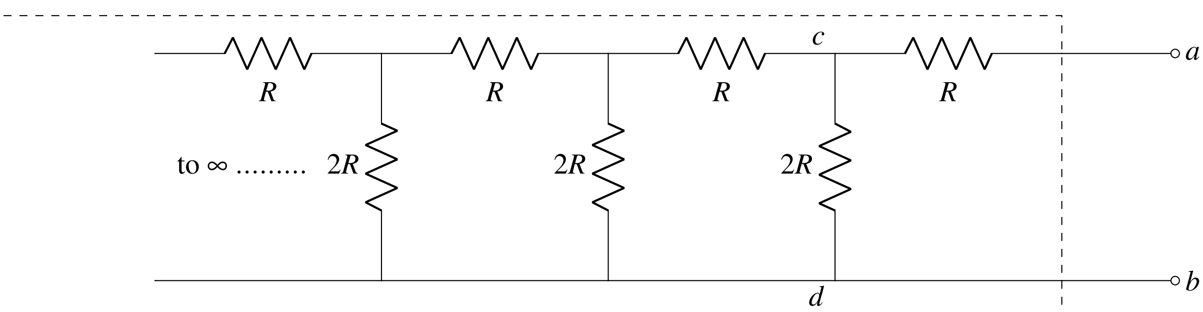




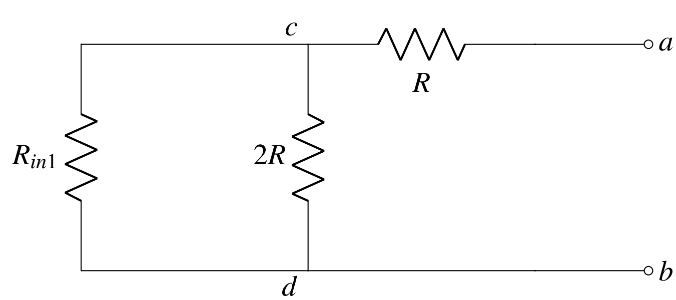


$$R_{eq} = 2R \parallel 4R = \frac{4}{3}R$$

(d) **(OPTIONAL, CHALLENGE)** Find the equivalent resistance for the infinite ladder looking in from points a and b. In other words, express the resistive network in the dashed region as one resistor. (*Hint: Let's call the resistance looking in from a and b as R_{in}, and the resistance looking to the left from points c and d as R_{in1}. Replace the entire circuit to the left of points c and d with a resistor whose value is given by R_{in1}. Find the relationship between R_{in} and R_{in1} using this circuit. Find another relationship between R_{in} and R_{in1} using the fact that the the ladder is infinite. For an infinite ladder, adding an another branch does not change the equivalent resistance. Think of this as a convergent infinite series.*)



As a first step you can replace the circuit looking to the left from c and d by R_{in1}.



Solution: We wish to compute the equivalent resistance R_{in} looking to the left from nodes a and b. The equivalent resistance looking to the left from nodes c and d is given by R_{in1}. Clearly,

$$R_{in} = (R_{in1}||2R) + R$$

Additionally, since this is an infinite ladder, the equivalent resistance does not change by addition of an extra branch to the right. Therefore, $R_{in1} = R_{in}$. Using this result in the previous equation, we have,

$$R_{in} = (R_{in}||2R) + R$$
$$R_{in} = \frac{2R \cdot R_{in}}{2R + R_{in}} + R$$
$$R_{in}^2 - RR_{in} - 2R^2 = 0$$
$$(R_{in} - 2R)(R_{in} + R) = 0$$

Clearly, $R_{in} = -R$ is not a physically realizable solution. The equivalent resistance looking into this infinite ladder is given by $R_{in} = 2R$.

3. Measuring Voltage and Current

Learning Goal: *The objective of this problem is to provide a deeper understanding in current and voltage measurement processes. It will also help you to understand how the electrical parameters of a measurement tool can affect the measurement precision.*

In order to measure quantities such as voltage and current, engineers use voltmeters and ammeters. A simple model of a voltmeter is a resistor with a very high resistance, R_{VM}. **The voltmeter measures the voltage across the resistance** R_{VM}. The measured voltage is then relayed to a microprocessor (such as the MSP430 microprocessor, which will be used in lab).

This model of an voltmeter is shown in Figure 1. Let us explore what happens when we connect this voltmeter to various circuits to measure voltages.
Throughout this problem assume $R_{\text{VM}} = 1\text{M}\Omega$. Recall that the SI prefix M or Mega is 10^6.

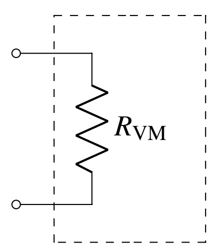


Figure 1: Our model of a voltmeter, $R_{\text{VM}} = 1\text{M}\Omega$

(a) Suppose we wanted to measure the voltage across R_2 (v_{out}) produced by the voltage divider circuit shown in Figure 2 on the left. The circuit on the right in Figure 2 shows how we would connect the voltmeter across R_2. Assume $R_1 = 100\Omega$ and $R_2 = 200\Omega$.
First calculate the value of v_{out}. Then calculate the voltage the voltmeter would measure, i.e. v_{meas}.

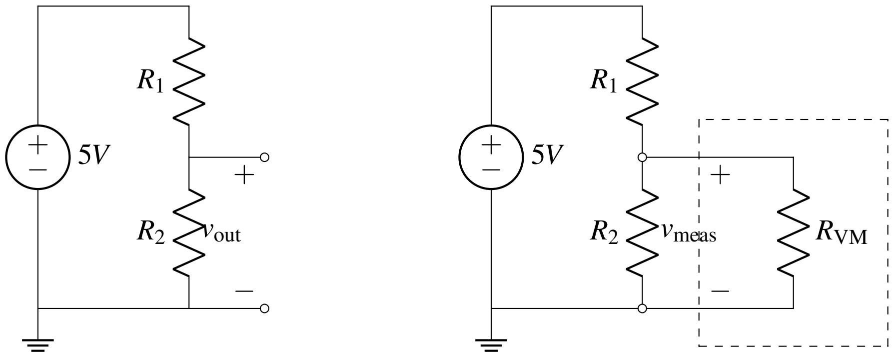


Figure 2: *Left:* Circuit without the voltmeter connected. *Right:* Voltmeter measuring voltage across R_2.

Solution: We start by finding v_{out} in the circuit on the left. Recognizing that this circuit is a voltage divider, we can directly find the following:

$$v_{\text{out}} = \frac{R_2}{R_1 + R_2} 5\text{V} = \frac{200\,\Omega}{300\,\Omega} 5\text{V} = 3.3333\text{V}$$

Next we consider the circuit on the right. We start by combining the resistor R_2 and R_{VM} since they are in parallel. Then we can apply the voltage divider formula to calculate the voltage across R_{VM}.

$$R_2 || R_{\text{VM}} = \frac{R_2 R_{\text{VM}}}{R_2 + R_{\text{VM}}} = \frac{200\,\Omega \cdot 1\,\text{M}\Omega}{200\,\Omega + 1\,\text{M}\Omega} = 199.96\,\Omega$$

$$v_{\text{meas}} = \frac{R_2 || R_{\text{VM}}}{R_1 + R_2 || R_{\text{VM}}} \cdot 5\,\text{V} = \frac{199.96\,\Omega}{100\,\Omega + 199.96\,\Omega} \cdot 5\,\text{V} = 3.3331\,\text{V}$$

(b) Repeat part (a), but now $R_1 = 10M\Omega$ and $R_2 = 10M\Omega$. Is this particular voltmeter still a good tool to measure the output voltage? Justify why or why not. (Notice that a *good* voltmeter should not significantly affect the value of voltages in a circuit by its presence.)

Solution: We start by again finding v_{out} in the circuit on the left. Recognizing that this circuit is a voltage divider, we can directly find the following:

$$v_{\text{out}} = \frac{R_1}{R_1 + R_2} 5\text{V} = \frac{10\,\text{M}\Omega}{20\,\text{M}\Omega} 5\text{V} = 2.5\text{V}$$

Next we consider the circuit on the right. We start by combining the resistor R_2 and R_{VM} since they are in parallel. Then we can apply the voltage divider formula to calculate the voltage across R_{VM}.

$$R_2 || R_{\text{VM}} = \frac{R_2 R_{\text{VM}}}{R_2 + R_{\text{VM}}} = \frac{10\,\text{M}\Omega \cdot 1\,\text{M}\Omega}{10\,\text{M}\Omega + 1\,\text{M}\Omega} = 0.909\,\text{M}\Omega$$

$$v_{\text{meas}} = \frac{R_2 || R_{\text{VM}}}{R_1 + R_2 || R_{\text{VM}}} \cdot 5\,\text{V} = \frac{0.909\,\text{M}\Omega}{10\,\text{M}\Omega + 0.909\,\text{M}\Omega} \cdot 5\,\text{V} = 0.4167\,\text{V}$$

Since the resistors R_1 and R_2 are larger than R_{VM}, using the voltmeter to measure element voltages significantly changes the value of V_{out}. Thus our voltmeter is not a good tool to use to measure the voltage for this circuit.

(c) Now suppose we are working with the same circuit as in part (a), but we know that $R_2 = R_1$. What is the maximum value of R_1 such that $v_{out} - v_{meas} \leq 0.1 \cdot v_{out}$ (i.e. v_{meas} is only 10% smaller than v_{out})?

Solution:

First, let's symbolically represent what the outputs are in the two cases. Recall that $R_1 = R_2$.

For the circuit without the voltmeter connected:

$$v_{\text{out}} = \frac{R_2}{R_1 + R_2} \cdot V_s = \frac{R_1}{R_1 + R_1} \cdot V_s = 0.5V_s$$

For the circuit with the voltmeter connected:

$$R_{\text{VM}}||R_2 = \frac{R_{\text{VM}}R_2}{R_{\text{VM}} + R_2}$$

$$v_{\text{meas}} = \frac{\frac{R_{\text{VM}}R_2}{R_{\text{VM}}+R_2}}{R_1 + \frac{R_{\text{VM}}R_2}{R_{\text{VM}}+R_2}} \cdot V_s = \frac{\frac{R_{\text{VM}}R_1}{R_{\text{VM}}+R_1}}{R_1 + \frac{R_{\text{VM}}R_1}{R_{\text{VM}}+R_1}} \cdot V_s$$

We can simplify the required inequality to:

$$v_{out} - v_{meas} \leq 0.1 \cdot v_{out} \Rightarrow 0.9v_{out} \leq v_{meas}$$

We also note that $R_1 \geq \frac{R_{\text{VM}}R_1}{R_{\text{VM}}+R_1}$ so that as we increase R_1, we end up reducing v_{meas}.

So the maximum value of R_1 will occur at the lowest permissible value of $v_{meas} = 0.9v_{out}$. We recall $R_{\text{VM}} = 1\,\text{M}\Omega$.

$$v_{meas} = 0.9v_{out}$$

$$\frac{\frac{R_{\text{VM}}R_1}{R_{\text{VM}}+R_1}}{R_1 + \frac{R_{\text{VM}}R_1}{R_{\text{VM}}+R_1}} \cdot V_s = 0.45V_s$$

$$\frac{\frac{1\,\text{M}\Omega \cdot R_1}{1\,\text{M}\Omega + R_1}}{R_1 + \frac{1\,\text{M}\Omega \cdot R_1}{1\,\text{M}\Omega + R_1}} = 0.45$$

$$\Rightarrow R_1 = 0.22\,\text{M}\Omega$$

(d) We can make **an ammeter** and measure the current through an element, using the combination of our voltmeter and an additional resistor R_x. The circuit shown in Figure 3 encompassed by the dashed box can work as an ammeter, where $R_x = 1\,\Omega$. We insert it in the circuit so that the current we want to measure flows through R_x, and then measure the current as $I_{\text{meas}} = \frac{V_{\text{VM}}}{R_x}$ where V_{VM} is the voltage across the voltmeter. $R_{VM} = 1\text{M}\Omega$ is the same as in previous parts.

In Figure 4, the voltmeter-resistor combo is connected so that the current we want to measure flows through resistor $R_1 = 1\text{k}\Omega$. For the circuit on the left, find the current through R_1 without the voltmeter-resistor combo connected (i.e. I_1). Then, for the circuit on the right, find the current measured by the voltmeter-resistor combo when it is connected as an ammeter (i.e. I_{meas}).

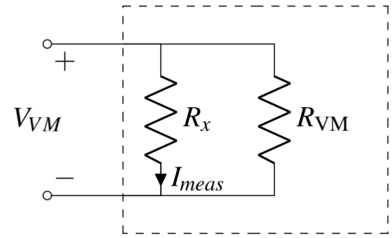


Figure 3: The voltmeter combined with resistor R_x to function as an ammeter (i.e. to measure current), $R_{\text{VM}} = 1M\Omega$.

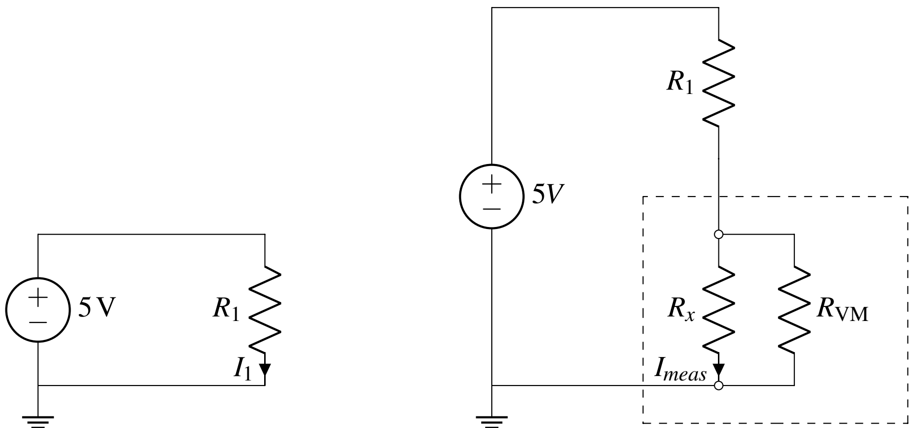


Figure 4: Circuits for part (d). *Left:* Original circuit; *Right:* Circuit with the voltmeter connected as an ammeter.

Solution:
We start with the circuit on the left

$$I_1 = \frac{5\,\text{V}}{1\,\text{k}\Omega} = 5\,\text{mA}$$

For the circuit on the right, we start by computing $R_x||R_{\text{VM}}$.

$$R_x||R_{\text{VM}} = \frac{R_x R_{\text{VM}}}{R_x + R_{\text{VM}}} = \frac{1\,\Omega \cdot 1\,\text{M}\Omega}{1\,\Omega + 1\,\text{M}\Omega} \approx 1\,\Omega$$

Next, we compute the voltage across the $R_x||R_{\text{VM}}$ combination. Notice this circuit is again a voltage divider.

$$V_{R_{\text{VM}}} = \frac{R_x||R_{\text{VM}}}{R_1 + R_x||R_{\text{VM}}} \cdot 5\,\text{V} = \frac{1\,\Omega}{1\,\text{k}\Omega + 1\,\Omega} \cdot 5\,\text{V} = 0.004\,995\,\text{V}$$

The measured current is this voltage divided by the resistance R_x.

$$I_{\text{meas}} = \frac{V_{R_{\text{VM}}}}{R_x} = \frac{0.004\,995\,\text{V}}{1\,\Omega} = 4.995\,\text{mA} \approx 5\,\text{mA}$$

(e) **(Optional)** What is the minimum value of R_1 that ensures the difference between current measurement (I_{meas}) and the the actual value (I_1) is such that $I_1 - I_{\text{meas}} \leq 0.1 \cdot I_1$, i.e. stays within $\pm 10\%$ of I_1? In other words, find the minimum allowable value for R_1 such that $I_1 - I_{\text{meas}} \leq 0.1 \cdot I_1$.

Hint: You can approximate $R_{\text{VM}}||R_x \approx R_x$ and $\frac{R_{\text{VM}}||R_x}{R_x} \approx 1$.

Solution:

Again, we will only consider the case where the measured current is smaller than the actual current, because the series combination of R_1 and $R_x||R_{\text{VM}}$ can only create a resistor bigger than R_1. First let's symbolically represent what the outputs are in the two cases:

For the circuit without the ammeter connected:

$$I_1 = \frac{V_s}{R_1}$$

For the circuit with the ammeter connected:

$$R_{\text{VM}}||R_x = \frac{R_{\text{VM}}R_x}{R_{\text{VM}}+R_x}$$
$$V_{\text{VM}} = \frac{\frac{R_{\text{VM}}R_x}{R_{\text{VM}}+R_x}}{R_1 + \frac{R_{\text{VM}}R_x}{R_{\text{VM}}+R_x}} \cdot V_s$$
$$I_{\text{meas}} = \frac{V_{\text{VM}}}{R_x} = \frac{\frac{R_{\text{VM}}}{R_{\text{VM}}+R_x}}{\frac{R_{\text{VM}}R_x}{R_{\text{VM}}+R_x} + R_1} \cdot V_s$$

In addition, the hint indicates we just need to consider the inequality:

$$I_1 - I_{\text{meas}} \le 0.1 \cdot I_1 \Rightarrow 0.9 I_1 \le I_{\text{meas}}$$

and that we can make the following approximations:

$$\frac{R_{\text{VM}}R_x}{R_{\text{VM}}+R_x} \approx R_x = 1\,\Omega$$
$$\frac{R_{\text{VM}}}{R_{\text{VM}}+R_x} \approx 1$$

This results in the following approximation for I_{meas}:

$$\begin{aligned} I_{\text{meas}} &= \frac{\frac{R_{\text{VM}}}{R_{\text{VM}}+R_x}}{\frac{R_{\text{VM}}R_x}{R_{\text{VM}}+R_x} + R_1} \cdot V_s \\ &\approx \frac{1}{R_x + R_1} \cdot V_s = \frac{1}{1\,\Omega + R_1} \cdot V_s \end{aligned}$$

Plugging in the expressions for I_1 and I_{meas}, we find:

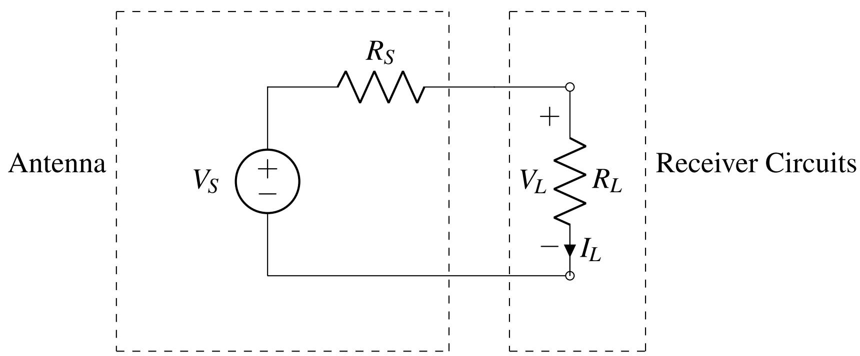
$$\begin{aligned} 0.9\frac{V_s}{R_1} &\le \frac{1}{1\,\Omega + R_1} \cdot V_s \\ \Rightarrow 0.9(1\,\Omega + R_1) &\le R_1 \\ \Rightarrow 0.9\,\Omega &\le 0.1 R_1 \\ \Rightarrow 9\,\Omega &\le R_1 \end{aligned}$$

So we find the minimum permissible value, $R_1 = 9\,\Omega$.

4. Maximum Power Transfer

Learning Goal: *This problem shows how the power dissipated in the load depends on the value of the load resistance. It also helps to understand the condition required for maximum power transfer.*

Smartphones use "bars" to indicate strength of the cellular signal. Fewer "bars" translate to slow or no connectivity. But what do these "bars" actually stand for? Voltage, current? Well, not quite. Good radio (a cellular modem is a particular type of radio) reception depends on the **power received at the receiver**. Communication theory tells us that higher received signal power enables higher data rates. To that end, we design a receiver that maximizes the power received, and hence connection speed. A typical receiver consists of an antenna and receiver circuits. The antenna receives the radio waves propagating in space, and converts it into electrical voltages and currents. A very good abstraction used by circuit designers is to **model the antenna as a voltage source V_S, with a series resistance R_S**. The typical values of V_S in a real cellular receiver are in the range of micro- or milli-volts (10^{-6} and 10^{-3}, respectively) and the typical values of resistance R_S are usually 50Ω or 75Ω, depending on how the antenna is designed. The receiver circuits are quite complex and will be covered in detail in EE142 "Integrated Circuits for Communications". However, a standard abstraction is to **model these receiver circuits as a load resistance R_L** to the antenna, as shown in the figure below.



Models are very important in engineering design for their ability to abstract away details when they are not needed and are the key to successful design of complex systems We will discuss the use and properties of electronic circuit models further in class.

Use the following component values for your calculations: $V_S = 100\mu V$, and $R_S = 50\Omega$.

(a) Consider any value of R_L within the range: $0 \leq R_L \leq \infty$. Find the value of R_L that maximizes the ***voltage*** V_L across resistor R_L. Calculate the values of V_L, I_L, and the power P_L dissipated by resistor R_L for the value you found.

(*Hint: The antenna voltage V_S and the resistance R_S are fixed. However, you are free to choose the value of R_L in order to maximize the voltage V_L. Alternatively, you may also intuitively argue for a particular value of R_L. How does the voltage across a resistor change as the value of the resistor increases?*)

Solution:
Note that this circuit is a voltage divider, where the voltage across R_L can be written as $V_L = V_S\left(\frac{R_L}{R_S+R_L}\right)$.
Taking the derivative of V_L with respect to R_L, we find $\frac{dV_L}{dR_L} = \frac{V_S R_S}{(R_S+R_L)^2}$. We note that as R_L increases, $\frac{dV_L}{dR_L}$ approaches 0.
Therefore, making R_L as large as possible (ideally $R_L = \infty$) maximizes V_L.
If $R_L = \infty$ then $V_L = V_S = 100\mu V$.

We can use the result $V_L = V_S\left(\frac{R_L}{R_S+R_L}\right)$ to find $I_L = \frac{V_L}{R_L} = \frac{V_S}{R_S+R_L}$. If $R_L = \infty$ then $I_L = 0\text{A}$.
We know that $P_L = V_L I_L$. Therefore, $P_L = (100\mu\text{V})(0\text{A}) = 0\text{W}$.

(b) Consider any value of R_L within the range: $0 \le R_L \le \infty$. Find the value of R_L that maximizes the ***current*** I_L through resistor R_L. Calculate the values of V_L, I_L, and the power P_L dissipated by resistor R_L for the value you found.

(*Hint: The antenna voltage V_S and the resistance R_S are fixed. However, you are free to choose the value of R_L in order to maximize the current I_L.*)

Solution:
We can again use the result $V_L = V_S\left(\frac{R_L}{R_S+R_L}\right)$ to calculate I_L and find what R_L should be to maximize I_L. We know that $I_L = \frac{V_L}{R_L} = \frac{V_S}{R_S+R_L}$. We can then see R_L should be as small as possible (ideally $R_L = 0$) to maximize I_L. If $R_L = 0$ then it behaves like a wire, and there will be no voltage drop across it. Therefore, $I_L = \frac{V_S}{R_S} = \frac{100\mu\text{V}}{50\Omega} = 2\mu\text{A}$. We know that $P_L = V_L I_L$. Therefore, $P_L = (0\text{V})(2\mu\text{A}) = 0\text{W}$.

(c) Find the value of R_L that maximizes the ***power*** P_L delivered to resistor R_L. Calculate the values of V_L, I_L, and the power P_L delivered to resistor R_L. **It is important to note that this value of R_L which maximizes the power delivered to R_L also optimizes cellular connectivity.** (*Hint: The power optimization is best performed algebraically by setting the derivative of P_L with respect to R_L to 0. Alternatively you can do the optimization graphically. Plot P_L versus R_L and find the maximum.*)

Solution: We can algebraically calculate the maximum of P_L by taking its derivative with respect to R_L. First, we write $P_L = I_L V_L = \left(\frac{V_S}{R_S+R_L}\right)\left(\frac{V_S R_L}{R_S+R_L}\right)$.

Next, we find that $\frac{dP_L}{dR_L} = \frac{{V_S}^2}{(R_S+R_L)^4}\left((R_S+R_L)^2 - 2R_L(R_S+R_L)\right)$.

We can then find when $\frac{dP_L}{dR_L} = 0$ by setting the expression $\left((R_S+R_L)^2 - 2R_L(R_S+R_L)\right)$ equal to 0.

We then find ${R_S}^2 + 2R_S R_L + {R_L}^2 - 2R_L R_S - 2{R_L}^2 = 0$. Further simplifying we find that ${R_S}^2 - {R_L}^2 = 0$.
This leads us to the final result that $R_L = R_S$ for maximizing the power in R_L.

If $R_L = 50\Omega$, then $V_L = 100\mu\text{V}\left(\frac{50\Omega}{(50\Omega+50\Omega)}\right) = 50\mu\text{V}$.

Using Ohm's Law, we know that $I_L = \frac{V_L}{R_L} = \frac{50\mu\text{V}}{50\Omega} = 1\mu\text{A}$.

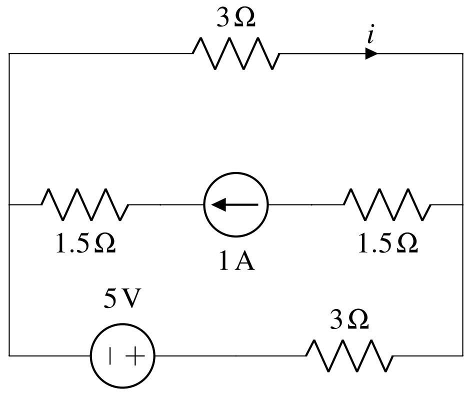
Finally, we know that $P_L = I_L V_L = (1\mu\text{A})(50\mu\text{V}) = 5\times10^{-11}\text{W}$.

To reiterate, this value of R_L which maximizes the power delivered to R_L also optimizes cellular connectivity. The next step is to design the receiver circuit such that it behaves like a resistor R_L and extracts the information received at the antenna. You will learn about this in detail in EE142.

5. Superposition

Learning Goal: *The objective of this problem is to help you practice solving circuits using the principles of superposition.*

Find the current i indicated in the circuit diagram below using superposition.

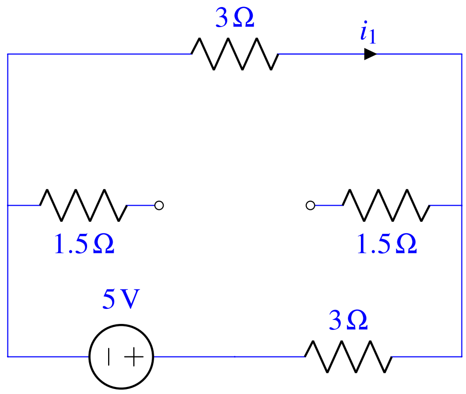


Solution:

$$i = -1/3\,\text{A}$$

Consider the circuits obtained by:

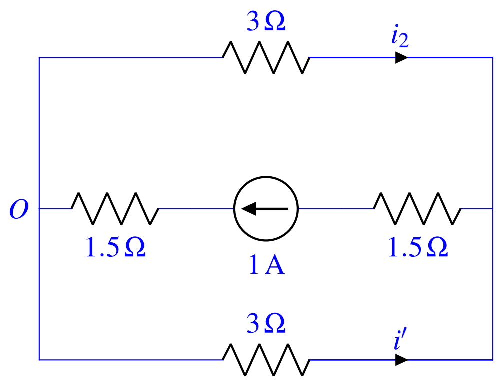
(a) Zeroing out the 1 A current source:



In the above circuit, no current flows through the middle branch, as it is an open circuit. Thus this is just a 5 V voltage source connected to two 3 Ω resistors in series so

$$i_1 = -5/6\,\text{A}$$

(b) Zeroing out the 5 V voltage source:



In the above circuit, notice that the $3\,\Omega$ resistors are in parallel and therefore form a current divider. Since the values of the resistances are equal, the current flowing through them will also be equal, that is $i_2 = i'$. Applying KCL to node O, we get

$$1 - i_2 - i' = 0$$

which gives us

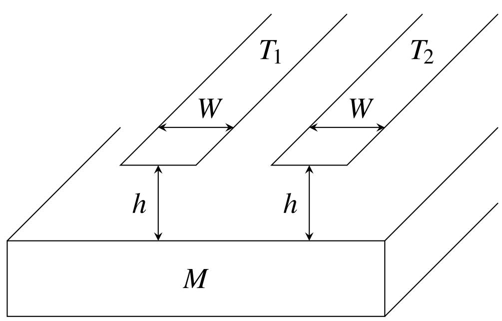
$$i_2 = 1/2\,\mathrm{A}$$

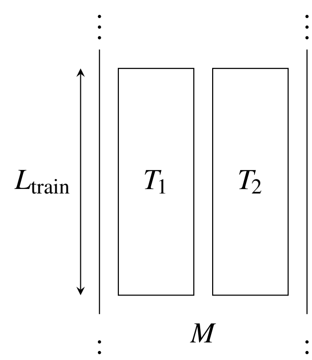
Now, applying the principle of superposition, we have $i = i_1 + i_2 = -5/6\,\mathrm{A} + 1/2\,\mathrm{A} = -1/3\,\mathrm{A}$.

6. Maglev Train Height Control System

One of the fastest forms of land transportation are trains that actually travel slightly elevated from ground using magnetic levitation (or "maglev" for short). Ensuring that the train stays at a relatively constant height above its "tracks" (the tracks in this case are what provide the force to levitate the train and propel it forward) is critical to both the safety and fuel efficiency of the train. In this problem, we'll explore how the maglev trains use capacitors to keep them elevated. (Note that real maglev trains may use completely different and much more sophisticated techniques to perform this function, so if you get a contract to build such a train, you'll probably want to do more research on the subject.)

(a) As shown below, let's imagine that all along the bottom of the train, we put two parallel strips of metal (T_1, T_2), and that on the ground below the train (perhaps as part of the track), we have one solid piece of metal (M).





Assuming that the entire train is at a uniform height above the track and ignoring any fringing fields (i.e., all capacitors are purely parallel plate), as a function of L_{train} (the length of the train), W (the width of T_1/T_2), and h (the height of the train off of the track), what is the capacitance between T_1 and M? How about the capacitance between T_2 and M?

Solution:

The distance between the plates (T_1 & M or T_2 & M) is h. The area of plate for the parallel plate capacitor is $A = WL_{\text{train}}$. Using the formula for capacitance of a parallel plate capacitor, we get:

$$C = \frac{\varepsilon A}{d}$$

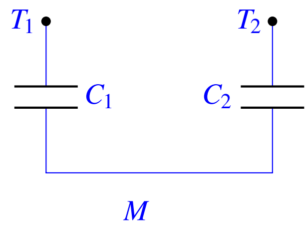
$$C_1 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between } \mathrm{T}_1 \text{ and M)}$$

$$C_2 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between } \mathrm{T}_2 \text{ and M)}$$

(b) Any circuit on the train can only make direct contact at T_1 and T_2. To detect the height of the train, it would only be able to measure the effective capacitance between T_1 and T_2. Draw a circuit model showing how the capacitors between T_1 and M and between T_2 and M are connected to each other.

Solution:

The capacitors C_1 and C_2 are in series. To realize this, let's consider the train circuit that is in contact with T_1 and T_2. If there is current entering plate T_1, the same current has to exit plate T_2. Thus, the circuit can be modeled as follows:



(c) Using the same parameters as in part (a), provide an expression for the capacitance between T_1 and T_2.

Solution:

Since the two capacitors are in series, the effective capacitance between T_1 and T_2 is given by:

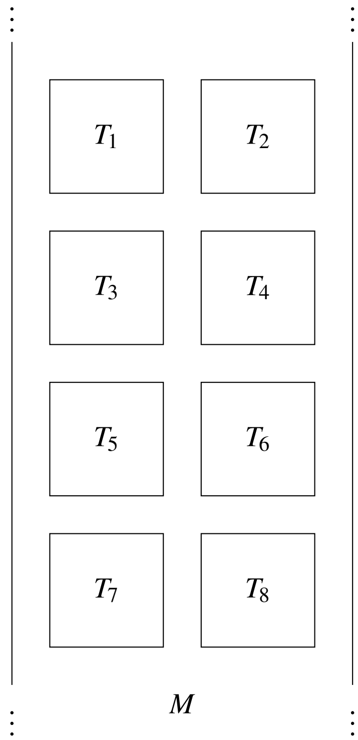
$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Thus, we get

$$\frac{1}{C_{\text{eq}}} = \frac{h}{\varepsilon W L_{\text{train}}} + \frac{h}{\varepsilon W L_{\text{train}}}$$

$$C_{\text{eq}} = \frac{\varepsilon W L_{\text{train}}}{2h}$$

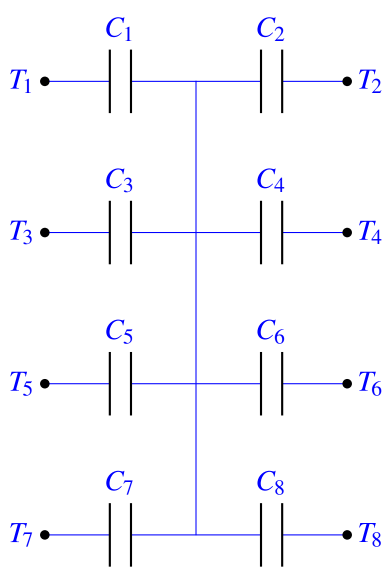
(d) So far we've assumed that the height of the train off of the track is uniform along its entire length, but in practice, this may not be the case. Suppose we modify the basic sensor design such that the two capacitive rails T_1 and T_2 are replaced with a grid of capacitors along the train's undercarriage, as illustrated in the diagram below. Please draw the equivalent circuit for this network.



Top View

Solution:

One important thing to note about this circuit is that it works only if extra care is taken during the capacitance measurement circuit. The equivalent model for this is:



7. (Optional/Practice) Cell Phone Battery

Learning Goal: *This problem explores how a battery can be modelled in a circuit. It also relates the concept of electric charge to current and energy.*

As great as smartphones are, one of their drawbacks is that their batteries don't last a long time. For example, a Google Pixel phone, under typical usage conditions (internet, a few cat videos, etc.), **uses 0.3W**. We will **model the battery as an ideal voltage source** (which maintains a constant voltage across its terminals regardless of current) except that we assume that **the voltage drops abruptly to zero when the battery is discharged** (in reality the voltage drops gradually, but let's keep things simple).

Battery capacity is specified in mAh, which indicates how many mA of current the battery can supply for one hour before it needs to be recharged. The Pixel's battery has a battery capacity of 2770mAh at 3.8V. For example, this battery could provide 1000mA (or 3.8W) for 2.77 hours before the voltage abruptly drops from 3.8V to zero.

(a) How long will a Pixel's full battery last under typical usage conditions? Remember that under typical usage conditions it uses 0.3W.

Solution:

300 mW of power at 3.8 V is about $\frac{300\,\text{mW}}{3.8\,\text{V}} = 79\,\text{mA}$ of current. Our 2770 mAh battery can supply 79 mA for $\frac{2770\,\text{mAh}}{79\,\text{mA}} = 35\,\text{h}$, or about a day and a half.

(b) How many coulombs of charge does the battery contain? How many usable electrons worth of charge are contained in the battery when it is fully charged? (An electron has 1.602×10^{-19} C of charge.)

Solution:

One hour has 3600 seconds, so the battery's capacity can be written as $2770\,\text{mAh} \times \frac{3600\,\text{s}}{1\,\text{h}} = 9.972 \times 10^{6}\,\text{mAs} = 9972\,\text{As} = 9972\,\text{C}$.

An electron has a charge of approximately 1.602×10^{-19} C, so $9972\,\text{C} \times \frac{1\,\text{electron}}{1.602 \times 10^{-19}\,\text{C}} \approx 6.225 \times 10^{22}$ electrons.

(c) Suppose the cell phone battery is completely discharged and you want to recharge it completely. How much energy (in J) is this? Recall that a J is equivalent to a W s.

Solution:

The battery capacity is 2770 mAh at 3.8 V, which means the battery has a total stored energy of $2770\,\text{mAh} \times 3.8\,\text{V} = 10.5\,\text{Wh}$. This is equal to $10.5\,\text{Wh} \times \frac{3600\,\text{s}}{1\,\text{h}} = 37.9\,\text{kJ}$.

(d) Suppose PG&E charges \$0.12 per kWh. Every day, you completely discharge the battery (meaning more than typical usage) and you recharge it every night. How much will recharging cost you for the month of October (31 days)?

Solution:

Discharging the battery once equates to $2770\,\text{mAh} \times 3.8\,\text{V} = 10.5\,\text{Wh}$, approximately 0.01 kWh. This costs $0.01\,\text{kWh} \times \frac{\$0.12}{1\,\text{kWh}} \times 31 = \$0.037$, or about 4 cents a month. Compare that to your cell phone data bill! Whew!

(e) The battery has internal circuitry that prevents it from getting overcharged (and possibly exploding!). We will model the battery and its internal circuitry as a resistor R_{bat}. We now wish to charge the battery by plugging into a wall plug. The wall plug can be modeled as a 5 V voltage source and 200 mΩ resistor, as pictured in Figure 5.

What is the power dissipated by R_{bat} for R_{bat}= 1 mΩ, 1 Ω, and 10 kΩ, i.e. how much power is being supplied to the phone battery as it is charging? How long will the battery take to charge for each value of R_{bat}?

Solution: The energy stored in the battery is 2770 mAh at 3.8 V, which is $2.77\,\text{Ah} \cdot 3.8\,\text{V} = 10.5\,\text{Wh}$. We can find the time to charge by dividing this energy by power in W to get time in hours.

We also note that the current (A) passing through R_{bat}, where R_{bat} is given in Ω, is:

$$I_{R_{\text{bat}}} = \frac{5}{R_{\text{bat}} + 0.2}$$

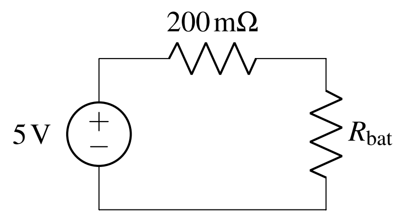


Figure 5: Model of wall plug, wire, and battery.

Therefore, the power (W) dissipated by R_{bat} is:

$$P_{R_{bat}} = I^2_{R_{bat}} R_{bat} = 25 \frac{R_{bat}}{(R_{bat} + 0.2)^2}$$

For $R_{bat} = 1\,\text{m}\Omega$:

$$P_{R_{bat}} = 25 \frac{R_{bat}}{(R_{bat} + 0.2)^2} = 25 \frac{0.001}{(0.001 + 0.2)^2} = 0.619\,\text{W}$$

The power dissipated by R_{bat} is $0.619\,\text{W}$, and the total time to charge the battery is $\frac{10.5\,\text{Wh}}{0.619\,\text{W}} = 17\,\text{h}$.

For $R_{bat} = 1\,\Omega$:

$$P_{R_{bat}} = 25 \frac{R_{bat}}{(R_{bat} + 0.2)^2} = 25 \frac{1}{(1 + 0.2)^2} = 17.36\,\text{W}$$

The power dissipated by R_{bat} is $17.36\,\text{W}$, and the total time to charge the battery is $\frac{10.5\,\text{Wh}}{17.36\,\text{W}} = 0.6\,\text{h}$, about $36\,\text{min}$.

For $R_{bat} = 10\,\text{k}\Omega$:

$$P_{R_{bat}} = 25 \frac{R_{bat}}{(R_{bat} + 0.2)^2} = 25 \frac{10^4}{(10^4 + 0.2)^2} = 2.5\,\text{mW}$$

The power dissipated by R_{bat} is $2.5\,\text{mW}$, and the total time to charge the battery is $\frac{10.5\,\text{Wh}}{0.0025\,\text{W}} = 4200\,\text{h}$.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.