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# EECS 16A    Designing Information Devices and Systems I

## Spring 2021    Homework 9

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**This homework is due April 2, 2021, at 23:59.**

**Self-grades are due April 5, 2021, at 23:59.**

### Submission Format

Your homework submission should consist of **one** file.

- `hw9.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

### 1. Reading Assignment

For this homework, please read Notes 16, 17 (17.1 - 17.2, specifically) and 17B. Note 16 will provide an introduction to capacitors (a circuit element which stores charge), capacitive equivalence, and the underlying physics behind them. Sections 17.1 - 17.2 in Note 17 will provide an overview of the capacitive touchscreen and how to measure capacitance. Note 17B will provide a walkthrough of the charge-sharing algorithm.

- Describe the key ideas behind how a capacitor works. How are capacitor equivalences calculated? Contrast this with how we calculate resistor equivalences.
- Consider the capacitive touchscreen. Describe how it works, and compare and contrast it to the resistive touchscreens we have seen in previous lectures and homeworks.
- What property of charge is applied in connecting phase 1 calculations to phase 2 calculations in the charge sharing algorithm?

#### Solution:

- A capacitor is a device that can store charge (and hence, energy) by separating two conducting surfaces with a non-conducting material. This allows equal and opposite amounts of charge to build up on both surfaces, creating a potential difference. Capacitors in parallel can be combined into an equivalent capacitance that is the sum of the individual capacitance (just like resistors in **series**). Capacitors  $C_1, C_2$  in series can be combined into an equivalent capacitance of  $C_1 || C_2$  (just like resistors in **parallel**).
- The capacitive touch screen works in short, by detecting a change in capacitance caused by the additional capacitance of a finger being added to the capacitance of the touch screen. The resistive touch screen on the other hand detects the position of a touch by modelling the touch screen as a voltage divider when pressed down.
- Conservation of charge at floating nodes allows us to relate measurements in different phases of a charge sharing protocol.

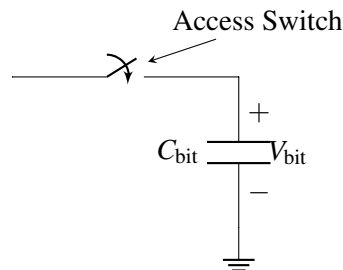
### 2. Dynamic Random Access Memory (DRAM)

Nearly all devices that include some form of computational capability (phones, tablets, gaming consoles, laptops, ...) use a type of memory known as Dynamic Random Access Memory (DRAM). DRAM is where

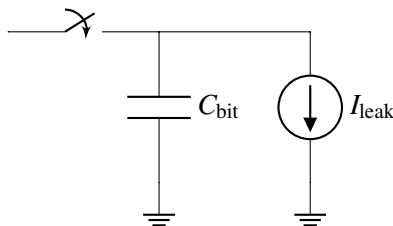
the “working set” of instructions and data for a processor is typically stored, and the ability to pack an ever increasing number of bits on to a DRAM chip at low cost has been critical to the continued growth in computational capability of our systems. For example, a single DRAM chip today can store  $> 8$  billion bits and is sold for  $\approx \$3$ - $\$5$ .

At the most basic level and as shown below, every bit of information that DRAM can store is associated with a capacitor. The amount of charge stored on that capacitor (and correspondingly, the voltage across the capacitor) determines whether a “1” or a “0” is stored in that location.

Single DRAM Bit Cell



In any real capacitor, there is always a path for charge to “leak” off the capacitor and cause it to eventually discharge. In DRAMs, the dominant path for this leakage to happen is through the access switch, which we will **model as a leakage to ground**. The figure below shows **a model of this leakage**:



*Fun Fact: This leakage is actually responsible for the “D” in “DRAM” – the memory is “dynamic” because after a cell is “written” by storing some charge onto its capacitor, if you leave the cell alone for too long, the value you wrote in will disappear because the charge on the capacitor leaked away.*

Let’s now try to use some representative numbers to compute how long a DRAM cell can hold its value before the information leaks away. Let  $C_{\text{bit}} = 28 \text{ fF}$  (note that  $1 \text{ fF} = 1 \times 10^{-15} \text{ F}$ ) and the capacitor be initially charged to  $1.2 \text{ V}$  to store a “1.”  $V_{\text{bit}}$  must be  $> 0.9 \text{ V}$  in order for the circuits outside of the column to properly read the bit stored in the cell as a “1.”

**What is the maximum value of  $I_{\text{leak}}$  that would allow the DRAM cell retain its value for  $> 1 \text{ ms}$ ?**

*Hint: Start by writing out the equations you know about charge and current related to the capacitor. Note here that the current source is discharging the capacitor.*

**Solution:**

We want the time that a cell can read a ‘1’ to be  $t_{\text{store}} = 1 \text{ ms}$ . We are given that  $V_{\text{init}} = 1.2 \text{ V}$  and that  $V_{\text{min}} = V_{\text{bit}} = 0.9 \text{ V}$ . To get an expression for the leakage rate, we differentiate

$$Q_{\text{bit}} = V_{\text{bit}} C_{\text{bit}},$$

giving

$$I_{\text{leak}} = \frac{dV_{\text{bit}}}{dt} C_{\text{bit}}.$$

Assuming a constant leakage rate, we have

$$\begin{aligned} I_{\text{leak}} &= \frac{\Delta V_{\text{bit}}}{\Delta t} C_{\text{bit}} \\ &= \frac{V_{\text{init}} - V_{\text{min}}}{t_{\text{store}}} C_{\text{bit}}. \end{aligned}$$

Plugging in the values from above, we get

$$\begin{aligned} I_{\text{leak}} &= \frac{(1.2 \text{ V} - 0.9 \text{ V}) \cdot 28 \text{ fF}}{1 \times 10^{-3} \text{ s}} \\ &= 8.4 \text{ pA}. \end{aligned}$$

### 3. Capacitive Touchscreen

The model for a capacitive touchscreen can be seen in Figure 1. See Table 1 for values of the dimensions. The green area represents the contact area of the finger with the top insulator. It has dimensions  $w_2 \times d_1$ , where  $w_2$  is the horizontal width of the finger contact area and  $d_1$  is the depth (into the page) of the finger contact area. The top metal (red area) has dimensions  $w_1 \times d_1$ . The bottom metal (grey area) has dimensions  $w \times d_2$ , where  $w$  is larger than both  $w_1$  and  $w_2$ . The vertical distance between the top metal (red) and bottom plate (grey) is  $t_1$ , and the vertical distance between the finger (green) and the bottom plate (grey) is  $t_2$ .

Table 1: Touchscreen Dimension Values

$d_1$	10 mm
$d_2$	1 mm
$t_1$	2 mm
$t_2$	4 mm
$w_1$	1 mm
$w_2$	2 mm

- (a) Draw the equivalent circuit of the touchscreen that contains the nodes  $F$ ,  $E_1$ , and  $E_2$  when: (i) there no finger present; and (ii) when there is a finger present. Express the capacitance values in terms of  $C_0$ ,  $C_{F-E_1}$ , and  $C_{F-E_2}$ .

*Hint: Note that node  $F$  represents the finger. When there is no touch node  $F$  would be non-existent. Hint: Treat  $E_1$  as the "top node",  $E_2$  as the "bottom node", and the finger  $F$  as an intermediate node when present.*

**Solution:**

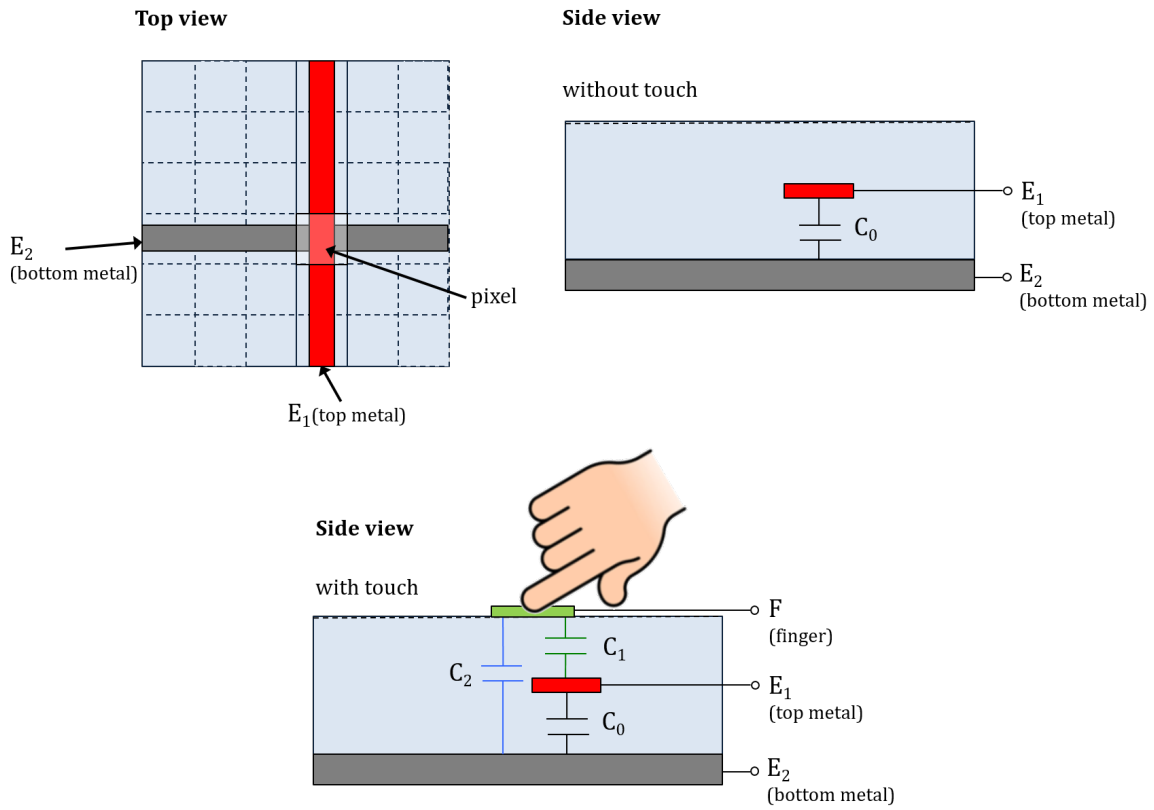


Figure 1: Model of capacitive touchscreen.

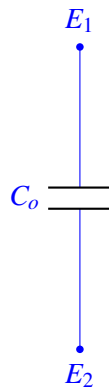


Figure 2: Touchscreen circuit with no finger present.

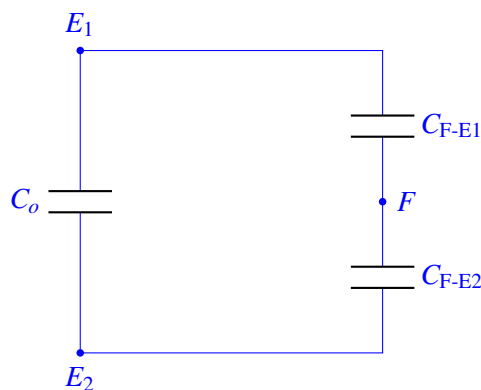


Figure 3: Touchscreen circuit with a finger present.

- (b) What are the values of  $C_0$ ,  $C_{E1-F}$ , and  $C_{F-E2}$ ? Assume that the insulating material has a permittivity of  $\epsilon = 4.43 \times 10^{-11} \text{ F/m}$  and that the thickness of the metal layers is small compared to  $t_1$  (so you can ignore the thickness of the metal layers). Also assume that the right edge of the top metal (red area) in the diagram is aligned with the right edge of the finger (green area) in the diagram. **Solution:**

$$C_0 = \epsilon \frac{d_2 w_1}{t_1} = 2.215 \times 10^{-14} \text{ F}$$

$$C_{F-E1} = \epsilon \frac{d_1 w_1}{t_2 - t_1} = 2.215 \times 10^{-13} \text{ F}$$

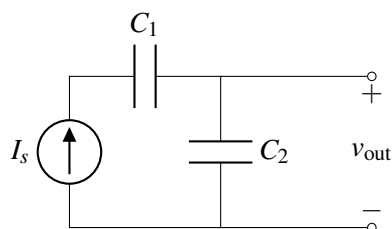
$$C_{F-E2} = \epsilon \frac{d_2 (w_2 - w_1)}{t_2} = 1.108 \times 10^{-14} \text{ F}$$

- (c) What is the difference in effective capacitance between the two metal plates (nodes  $E_1$  and  $E_2$ ) when a finger is present?

**Solution:** The effective capacitance between the two plates is  $C_0 = 2.215 \times 10^{-14} \text{ F}$  when there is no finger. When there is a finger, we have  $C_0$  in parallel with a series combination of  $C_{F-E1}$  and  $C_{F-E2}$ , giving an additional capacitance  $C_{F-E1} || C_{F-E2} = 1.055 \times 10^{-14} \text{ F}$  when a finger is present. Therefore, the total effective capacitance is:  $3.270 \times 10^{-14} \text{ F}$ .

#### 4. More Current Sources And Capacitors

- (a) For the circuit given below, give an expression for  $v_{\text{out}}(t)$  in terms of  $I_s$ ,  $C_1$ ,  $C_2$ , and time  $t$ . Assume that all capacitors are initially uncharged, i.e. the initial voltage across each capacitor is 0V.



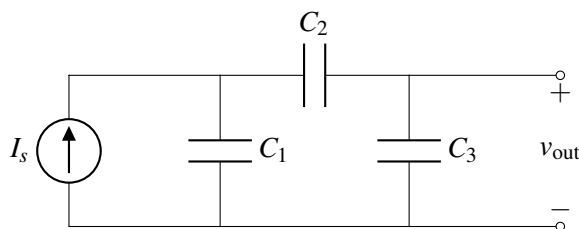
**Solution:**

By KCL, the current  $I_s$  flowing through  $C_1$  must be the current flowing through  $C_2$ .  $v_{\text{out}}(0) = 0$  because all capacitors are initially uncharged.

$$I_s = C_2 \frac{dv_{\text{out}}(t)}{dt}$$

$$v_{\text{out}}(t) = \int_0^t \frac{I_s}{C_2} d\tau = \frac{I_s}{C_2} t + v_{\text{out}}(0) = \frac{I_s}{C_2} t$$

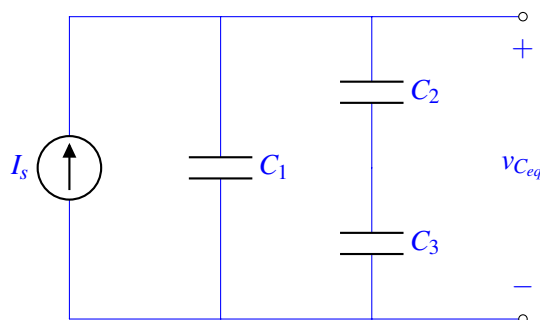
- (b) **(Practice)** For the circuit given below, give an expression for  $v_{\text{out}}(t)$  in terms of  $I_s$ ,  $C_1$ ,  $C_2$ ,  $C_3$ , and  $t$ . Assume that all capacitors are initially uncharged, i.e. the initial voltage across each capacitor is 0V. You may choose to use either NVA or capacitor equivalences to help you solve this problem.

**Solution:**

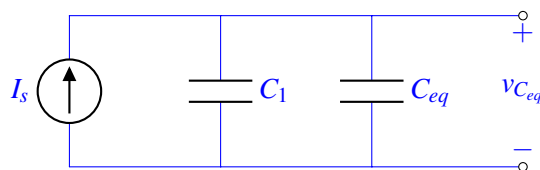
We will present two methods of calculating  $v_{\text{out}}(t)$ : using capacitor equivalences as well as our tried-and-true node voltage analysis technique.

**1. Capacitor Equivalences**

Instead of finding  $v_{\text{out}}$  directly, let's first find the voltage  $v_{C_{eq}}$  across  $C_2$  and  $C_3$ .



To do this, we recognize that  $C_2$  and  $C_3$  are in series. We can therefore replace them with their equivalent capacitance  $C_{eq} = C_2 \parallel C_3 = \frac{C_2 C_3}{C_2 + C_3}$ .



We know that to solve for  $v_{C_{eq}}$ , we can find the equivalent capacitance of  $C_1$  and  $C_{eq}$  first. Since they are in parallel, we combine them as  $C_1 + C_{eq}$ . Since all the capacitors are initially uncharged,  $v_{C_{eq}}(0) = 0$ .

$$v_{C_{eq}}(t) = \int_0^t \frac{I_s}{C_1 + C_{eq}} d\tau = \frac{I_s}{C_1 + C_{eq}} t + v_{C_{eq}}(0) = \frac{I_s}{C_1 + C_{eq}} t$$

Now that we know that voltage across the equivalent capacitor  $C_{eq}$ , we can find the current flowing through the equivalent capacitor  $C_{eq}$ .

$$i_{C_{eq}}(t) = C_{eq} \frac{dv_{C_{eq}}(t)}{dt} = C_{eq} \frac{I_s}{C_1 + C_{eq}}$$

Note that the current  $i_{C_{eq}}$  is equal to the current flowing through  $C_3$  since  $C_2$  and  $C_3$  were originally connected in series.

$$i_{C_3}(t) = i_{C_{eq}}(t) = C_{eq} \frac{I_s}{C_1 + C_{eq}}$$

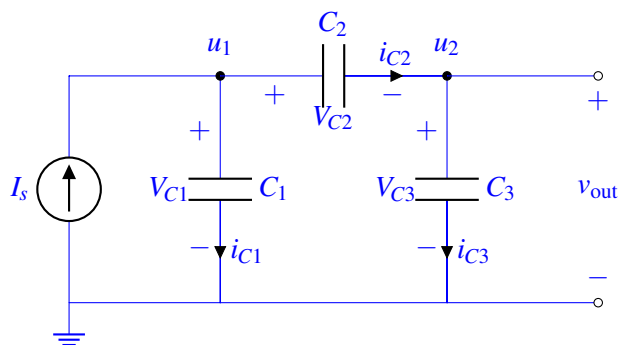
Since  $v_{out}$  is the voltage across the capacitor  $C_3$ , we integrate to find  $v_{out}$ . Again, since all capacitors are initially uncharged,  $v_{out}(0) = 0$ .

$$i_{C_3}(t) = C_3 \frac{dv_{out}(t)}{dt}$$

$$v_{out}(t) = \int_0^t \frac{1}{C_3} C_{eq} \frac{I_s}{C_1 + C_{eq}} d\tau = \frac{C_{eq} I_s}{C_3 (C_1 + C_{eq})} t + v_{out}(0) = \frac{\frac{C_2 C_3}{C_2 + C_3} I_s t}{C_3 \left( C_1 + \frac{C_2 C_3}{C_2 + C_3} \right)} t = \frac{C_2 I_s}{C_1 C_2 + C_1 C_3 + C_2 C_3} t$$

## 2. Node Voltage Analysis

We begin by labeling our circuit in its entirety:



Writing KCL equations at  $u_1$  and  $u_2$  gives the following two equations:

$$u_1 : I_s = i_{C1} + i_{C2}$$

$$u_2 : i_{C2} = i_{C3}$$

Recall that differentiating the capacitor-voltage-charge relationship  $Q = CV$  with respect to time gives us  $\frac{dQ}{dt} = I = C \cdot \frac{dV}{dt}$ . We can apply this result to our KCL equations to first solve for the *change* in  $v_{out}$ , i.e.,  $\frac{dv_{out}}{dt} = \frac{du_2}{dt}$ , and then in the end integrate this quantity to find our final equation  $v_{out}(t)$ .

Rewriting our KCL equations, we obtain:

$$u_1 : I_s = C_1 \frac{du_1}{dt} + C_2 \frac{d(u_1 - u_2)}{dt}$$

$$u_2 : C_2 \frac{d(u_1 - u_2)}{dt} = C_3 \frac{du_2}{dt}$$

Let's rearrange the KCL equation for node  $u_2$  to solve for  $\frac{du_2}{dt}$ .

$$C_2 \frac{du_1}{dt} = (C_2 + C_3) \frac{du_2}{dt}$$

$$\implies \frac{du_2}{dt} = \frac{C_2}{C_2 + C_3} \frac{du_1}{dt}$$

We then plug this result into the first KCL equation written at  $u_1$  to solve for  $\frac{du_1}{dt}$ :

$$I_s = C_1 \frac{du_1}{dt} + C_2 \frac{du_1}{dt} - \frac{C_2^2}{C_2 + C_3} \frac{du_1}{dt}$$

$$= \left( C_1 + C_2 - \frac{C_2^2}{C_2 + C_3} \right) \frac{du_1}{dt}$$

$$= \left( \frac{C_1(C_2 + C_3) + C_2(C_2 + C_3) - C_2^2}{C_2 + C_3} \right) \frac{du_1}{dt}$$

$$\implies \frac{du_1}{dt} = I_s \cdot \frac{C_2 + C_3}{C_1 C_2 + C_1 C_3 + C_2 C_3}$$

Finally, we can substitute this result back into our result for  $\frac{du_2}{dt}$ , which we stated earlier is equivalent to  $\frac{dV_{out}}{dt}$ :

$$\frac{dV_{out}}{dt} \equiv \frac{du_2}{dt} = \frac{C_2}{C_2 + C_3} \frac{du_1}{dt}$$

$$= \frac{C_2}{C_2 + C_3} \cdot I_s \cdot \frac{C_2 + C_3}{C_1 C_2 + C_1 C_3 + C_2 C_3}$$

$$= I_s \cdot \frac{C_2}{C_1 C_2 + C_1 C_3 + C_2 C_3}$$

Lastly, we integrate  $\frac{dV_{out}}{dt}$  and get our final  $v_{out}(t)$  expression, which should match what was calculated using the capacitor equivalences technique.

$$v_{out}(t) = \int_0^t \frac{dV_{out}}{dt} d\tau$$

$$= \int_0^t I_s \cdot \frac{C_2}{C_1 C_2 + C_1 C_3 + C_2 C_3} d\tau$$

$$= \frac{I_s C_2}{C_1 C_2 + C_1 C_3 + C_2 C_3} t$$

## 5. Circuit with Capacitors

Find the voltages at nodes A and B, and currents flowing through all of the capacitors at steady state. Assume that before the voltage source is applied, the capacitors all initially have a charge of 0 Coulombs.

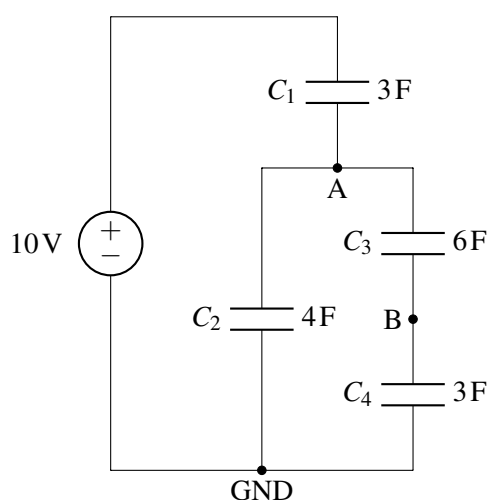
In general your strategy to solve circuits with capacitors should be similar to solving resistive circuits. For capacitive circuits we often care about steady state (i.e. what happens to the circuit after a long time and no more changes are happening). If we are considering a circuit with capacitors and voltage sources, we will always be thinking about steady state (or the steady state for a phase if we are doing a charge sharing problem with switches). When thinking about steady state you always want to write out the equations for charge that you know, as well as all the KVL type relationships around voltages you know. Then use the key



idea that charge is conserved to build out your system of equations. Don't be daunted by the variable names and know that everything just boils down to a system of linear equations.

Here are some principles that are also helpful:

- Charge at a node from which charge cannot escape or enter (floating node) is always conserved. — if the sum of charges is 0 on a floating node, the sum of charges on that floating node at steady state will be zero.
- The charge  $Q$  stored in a capacitor is given by the equation  $Q = CV$ . That is, the plate that corresponds to the "+" terminal, stores  $+Q = +CV$ , and the plate that corresponds to the "-" terminal, stores  $-Q = -CV$ .
- If two capacitors are initially uncharged, and then are connected in series, the charges on both capacitors are equal to each other at steady state.
- The voltage across capacitors in parallel is equal at steady state.



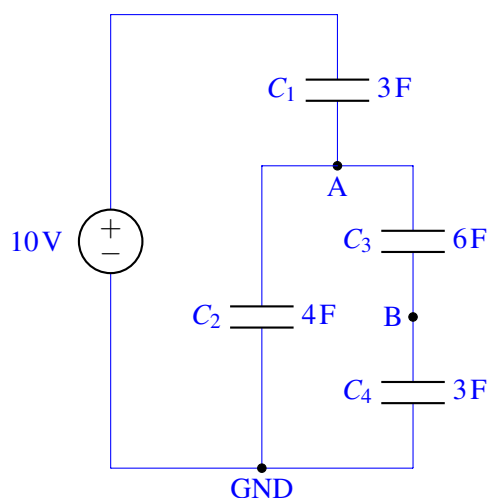
### Solution:

#### Method 1: Charge conservation

For a capacitor  $C_k$ , let us denote the voltage across it by  $v_{C_k}$ , the current flowing through it by  $i_{C_k}$ , and its charge by  $Q_{C_k}$ . In steady state (that is, after the current has been running for a very long time), direct current (DC) capacitors act as open circuits. Hence, we see that there is no current flowing through the capacitors, that is,

$$i_{C_1} = i_{C_2} = i_{C_3} = i_{C_4} = 0 \text{ A.}$$

For finding the voltages across the capacitors, let us label nodes on the circuit as shown in the following figure.



We are going to use the following four properties to find the voltages across the capacitors:

- Charge is always conserved at floating nodes.
- The charge  $Q$  stored in a capacitor is given by the equation  $Q = CV$ .
- The charges across series capacitors that are initially uncharged are equal to each other.
- The voltage across parallel capacitors is equal.

As an example use of property (c), we have the charge on the capacitor  $C_3$  equal to the charge on the capacitor  $C_4$ .

Let us start by writing the equation for conservation of charge at node A:

$$Q_{C_1} = Q_{C_2} + Q_{C_3}$$

By property (b), that is,  $Q = CV$ , we can equivalently write this equation for charge conservation in terms of node voltages as

$$(10\text{V} - v_A)3\text{F} = v_A 4\text{F} + (v_A - v_B)6\text{F},$$

which, after simplifying the equation, gives

$$30\text{V} = 13v_A - 6v_B. \quad (1)$$

Let us then write the charge conservation equation at node B; we have

$$Q_{C_3} = Q_{C_4}.$$

As before, we can write this charge conservation equation in terms of the node voltages as

$$(v_A - v_B)6\text{F} = v_B 3\text{F},$$

which, after simplification, gives

$$2v_A = 3v_B. \quad (2)$$

Equations 1 and 2 give us two linearly independent equations in two unknowns. Solving the system, we get

$$\begin{aligned} v_A &= 10/3\text{V}, \\ v_B &= 20/9\text{V}. \end{aligned}$$

Using the node voltages, we can calculate the voltages across the capacitors as

$$v_{C_1} = 10\text{ V} - v_A = 20/3\text{ V},$$

$$v_{C_2} = v_A = 10/3\text{ V},$$

$$v_{C_3} = v_A - v_B = 10/9\text{ V},$$

$$v_{C_4} = v_B = 20/9\text{ V}.$$

We write the currents across the capacitors again here for reader's convenience:

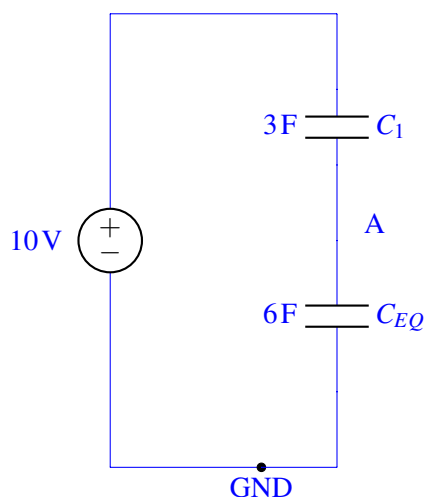
$$i_{C_1} = i_{C_2} = i_{C_3} = i_{C_4} = 0\text{ A}$$

### Method 2: Capacitor equivalence

Let's try to consider another method of solving this. We know that, initially, all the capacitors have charges of 0 C. After a 10 V voltage source is applied, the intermediate node voltages  $v_A$  and  $v_B$  will settle to some steady-state value.

Note that capacitor voltage division only works here because we know the initial conditions of the capacitors before and after the 10 V voltage source is applied. Capacitor voltage division is really just another way of solving for charge redistribution.

Let's try to find the voltage  $v_A$ .



Note that we replaced the capacitors below node A with an equivalent capacitance  $C_{EQ} = (6\text{ F} \parallel 3\text{ F}) + 4\text{ F} = 6\text{ F}$  (can you prove to yourself why?). The equation for  $v_A$  uses the *capacitor* voltage division formula:

$$v_A = 10\text{ V} \frac{3\text{ F}}{3\text{ F} + 6\text{ F}} = 10/3\text{ V}$$

We can then recognize that the voltage  $v_B$  is the capacitor voltage division of  $v_A$ , namely:

$$\begin{aligned} v_B &= \frac{6\text{ F}}{6\text{ F} + 3\text{ F}} v_A \\ &= \left(\frac{2}{3}\right) 10/3\text{ V} \\ &= 20/9\text{ V} \end{aligned}$$

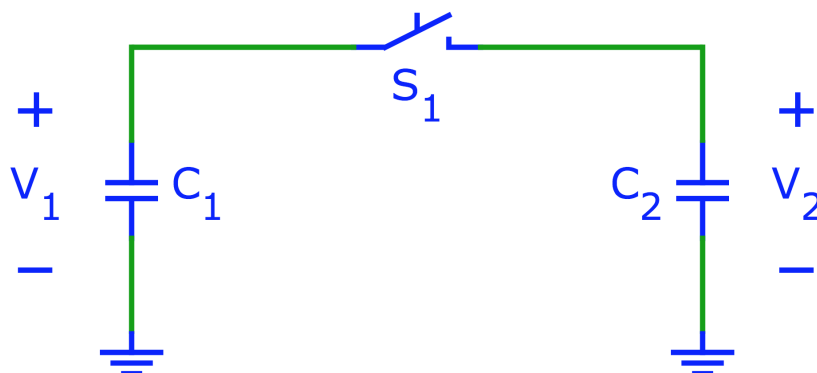
Note that these are the same values we found using Method 1.

## 6. Charge Sharing

In the circuit below, switch  $S_1$  is initially open. Capacitor  $C_1 = 10^{-3}\text{F}$  is initially charged to  $V_1 = 3\text{V}$  and capacitor  $C_2 = 3 \times 10^{-3}\text{F}$  is initially charged to  $V_2 = 2\text{V}$ .

Now  $S_1$  is closed. Calculate the new value of  $V_2$ .

*Hint:* Remember that charge is conserved at floating nodes.



### Solution:

Let us define the initial charge on  $C_1$  as  $Q_{1i}$  and the initial charge on  $C_2$  as  $Q_{2i}$ . We know that  $Q_{1i} = C_1 V_{1i}$  and  $Q_{2i} = C_2 V_{2i}$ , where  $V_{1i}$  and  $V_{2i}$  are the initial voltages across  $C_1$  and  $C_2$ , respectively. (i.e. before switch  $S_1$  is closed). We know from conservation of charge that  $Q_{1i} + Q_{2i} = Q_{1f} + Q_{2f}$ , where  $Q_{1f}$  and  $Q_{2f}$  are the final charge on  $C_1$  and  $C_2$ . (i.e. after switch  $S_1$  is closed). We can write this as:

$$(1) \quad C_1 V_{1i} + C_2 V_{2i} = Q_{1f} + Q_{2f}.$$

Additionally, we know that once switch  $S_1$  is closed, the voltage across  $C_1$  and  $C_2$  must be the same, because they are now in parallel with each other. Specifically,  $V_{1f} = V_{2f}$  where  $V_{1f}$  and  $V_{2f}$  are the final voltages across  $C_1$  and  $C_2$ , respectively. (i.e. after switch  $S_1$  is closed).

We can write this as:

$$(2) \quad V_{1f} = \frac{Q_{1f}}{C_1} = V_{2f} = \frac{Q_{2f}}{C_2}.$$

Putting equations (1) and (2) into matrix form, we get the following:

$$\begin{bmatrix} 1 & 1 \\ \frac{1}{C_1} & \frac{-1}{C_2} \end{bmatrix} \begin{bmatrix} Q_{1f} \\ Q_{2f} \end{bmatrix} = \begin{bmatrix} C_1 V_{1i} + C_2 V_{2i} \\ 0 \end{bmatrix}.$$

Plugging in the numbers given to us in the problem we get:

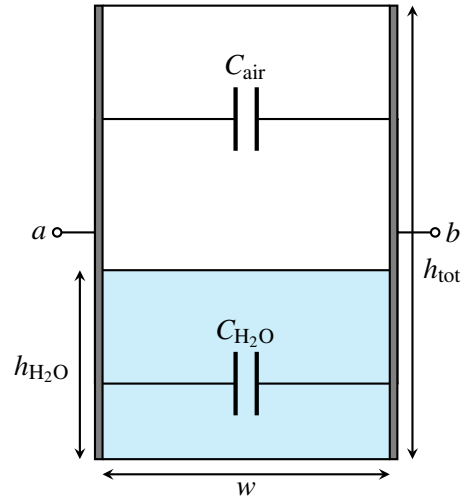
$$\begin{bmatrix} 1 & 1 \\ \frac{1}{0.001} & \frac{-1}{0.003} \end{bmatrix} \begin{bmatrix} Q_{1f} \\ Q_{2f} \end{bmatrix} = \begin{bmatrix} (0.001)(3) + (0.003)(2) \\ 0 \end{bmatrix}.$$

Using either back substitution, Gaussian elimination, or iPython, we find that  $Q_{1f} = 0.00225\text{C}$  and  $Q_{2f} = 0.00675\text{C}$ . We can then calculate  $V_{2f} = \frac{Q_{2f}}{C_2} = \frac{0.00675\text{C}}{0.003\text{F}} = 2.25\text{V}$ .

(Also, note that  $V_{1f} = \frac{Q_{1f}}{C_1} = \frac{0.00225\text{C}}{0.001\text{F}} = 2.25\text{V}$ .)

## 7. It's finally raining!

A lettuce farmer in Salinas Valley has grown tired of weather.com's imprecise rain measurements. Therefore, they decided to take matters into their own hands by building a rain sensor. They placed a rectangular tank outside and attached two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside.



The width and length of the tank are both  $w$  (i.e., the base is square) and the height of the tank is  $h_{\text{tot}}$ .

- (a) What is the capacitance between terminals  $a$  and  $b$  when the tank is full? What about when it is empty?  
*Note:* the permittivity of air is  $\epsilon$ , and the permittivity of rainwater is  $81\epsilon$ .

**Solution:**

Capacitance of parallel plates is governed by the equation:

$$C = \frac{\epsilon A}{d},$$

where  $\epsilon$  is the *permittivity* of the dielectric material,  $A$  is the area of the plates, and  $d$  is the distance between the plates. If we apply this to our physical structure, we find that the area of the plates are  $h_{\text{tot}} \cdot w$ , and the distance between the plates is  $w$ . The only difference here between a full and empty tank is the permittivity of the material between the two plates.

$$C_{\text{empty}} = \frac{\epsilon_{\text{air}} h_{\text{tot}} w}{w} = \epsilon h_{\text{tot}}$$

$$C_{\text{full}} = \frac{\epsilon_{\text{H}_2\text{O}} h_{\text{tot}} w}{w} = 81\epsilon h_{\text{tot}}$$

- (b) Suppose the height of the water in the tank is  $h_{\text{H}_2\text{O}}$ . Modeling the tank as a pair of capacitors in parallel, find the total capacitance between the two plates. Call this capacitance  $C_{\text{tank}}$ .

**Solution:**

We can break the total capacitance into two parts. First, let's calculate the capacitance of the two plates separated by water:

$$C_{\text{water}} = \frac{\epsilon_{\text{H}_2\text{O}} h_{\text{H}_2\text{O}} w}{w} = 81\epsilon h_{\text{H}_2\text{O}}$$

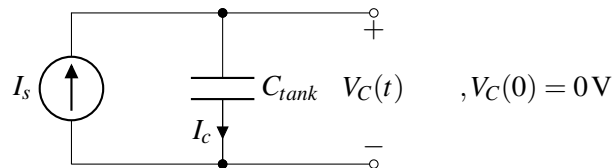
And now we can calculate the capacitance of the two plates separated by air:

$$C_{\text{air}} = \frac{\epsilon_{\text{air}}(h_{\text{tot}} - h_{\text{H}_2\text{O}})w}{w} = \epsilon(h_{\text{tot}} - h_{\text{H}_2\text{O}})$$

Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

$$C_{\text{tank}} = C_{\text{water}} + C_{\text{air}} = \epsilon(h_{\text{tot}} + 80h_{\text{H}_2\text{O}})$$

- (c) After building this capacitor, the farmer consults the internet to assist them with a capacitance-measuring circuit. A fellow internet user recommends the following:



In this circuit,  $C_{\text{tank}}$  is the total tank capacitance that you calculated earlier.  $I_s$  is a known current supplied by a current source.

The suggestion is to measure  $V_C$  for a brief interval of time, and then use the difference to determine  $C_{\text{tank}}$ .

Determine  $V_C(t)$ , where  $t$  is the number of seconds elapsed since the start of the measurement. You should assume that before any measurements are taken, the voltage across  $C_{\text{tank}}$ , i.e.  $V_C$ , is initialized to 0V, i.e.  $V_C(0) = 0$ .

**Solution:** The element equation for the capacitor is:

$$I_C = C_{\text{tank}} \frac{dV_C}{dt}$$

We also know from KCL that:

$$I_C = I_s$$

Thus, we get the following differential equation for  $V_C$ :

$$\frac{dV_C}{dt} = \frac{I_s}{C_{\text{tank}}}$$

We recall that  $I_s$  and  $C_{\text{tank}}$  are constant values and the initial value of  $V_C$  is zero ( $V_C(0) = 0$ ). Applying these facts and integrating the differential equation, we get the following equation for  $V_C$ :

$$V_C(t) = \frac{I_s}{C_{\text{tank}}}t$$

- (d) Using the equation you derived for  $V_C(t)$ , describe how you can use this circuit to determine  $C_{\text{tank}}$  and  $h_{\text{H}_2\text{O}}$ .

**Solution:** We connect the current source providing  $I_s$  A to the capacitor  $C_{\text{tank}}$ . At the same time, we can measure  $V_C(t)$ . After some time passes, we measure  $V_C(t)$  and plug it into the following equation (assuming, as before, that  $V_C(0) = 0$ ):

$$C_{tank} = \frac{I_s}{V_C(t)} t$$

If we know  $C_{tank}$ , we can determine  $h_{H_2O}$ . Using the equation derived in part (b), we see that

$$h_{H_2O} = \frac{C_{tank} - h_{tot}\epsilon}{80\epsilon}$$

## 8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

### **Solution:**

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.